Stanford Mathematics Tournament 2021

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by parmenides51

– Team Round

– **p1.** What is the length of the longest string of consecutive prime numbers that divide 224444220?

p2. Two circles of radius r are spaced so their centers are $2r$ apart. If $A(r)$ is the area of the smallest square containing both circles, what is $\frac{A(r)}{r^2}$?

p3. Define $e_0 = 1$ and $e_k = e^{e_{k-1}}$ for $k \geq 1$. Compute $\int_{e_4}^{e_6} \frac{1}{x(\ln x)(\ln \ln x)(\ln \ln \ln x)}dx$.

p4. Compute $\prod_{n=0}^{2020} \sum_{k=0}^{2020} \left(\frac{1}{2020}\right)^{k(2021)^n}$

p5. A potter makes a bowl, beginning with a sphere of clay and then cutting away the top and bottom of the sphere with two parallel cuts that are equidistant from the center. Finally, he hollows out the remaining shape, leaving a base at one end. Assume that the thickness of the bowl is negligible. If the potter wants the bowl to hold a volume that is $\frac{13}{27}$ of the volume of the sphere he started with, the distance from the center at which he makes his cuts should be what fraction of the radius?

p6. Let AB be a line segment with length $2 + \sqrt{2}$. A circle ω with radius 1 is drawn such that it passes through the end point B of the line segment and its center O lies on the line segment AB. Let C be a point on circle ω such that $AC = BC$. What is the size of angle CAB in degrees?

p7. Find all possible values of $\sin x$ such that $4 \sin(6x) = 5 \sin(2x)$.

p8. Frank the frog sits on the first lily pad in an infinite line of lily pads. Each lily pad besides the one first one is randomly assigned a real number from 0 to 1. Franks starting lily pad is assigned 0. Frank will jump forward to the next lily pad as long as the next pad's number is greater than his current pad's number. For example, if the first few lily pads including Frank's are numbered $0, .4, .72, .314$, Frank will jump forward twice, visiting a total of 3 lily pads. What is the expected number of lily pads that Frank will visit?

p9. For positive integers n and k with $k \leq n$, let

$$f(n,k) = \sum_{j=0}^{k-1} j \binom{k-1}{j} \binom{n-k+1}{k-j}.$$

Compute the sum of the prime factors of $f(4,4) + f(5,4) + f(6,4) + ... + f(2021,4)$.

p10. $\triangle ABC$ has side lengths $AB = 5$, $AC = 10$, and $BC = 9$. The median of $\triangle ABC$ from A intersects the circumcircle of the triangle again at point D. What is $BD + CD$?

p11. A subset of five distinct positive integers is chosen uniformly at random from the set $\{1, 2, ..., 11\}$. The probability that the subset does not contain three consecutive integers can be written as m/n, where m and n are relatively prime positive integers. Find $m + n$.

p12. Compute $\int_{-1}^{1}(x^2 + x + \sqrt{1 - x^2})^2 dx$.

p13. Three friends, Xander, Yulia, and Zoe, have each planned to visit the same cafe one day. If each person arrives at the cafe at a random time between 2 PM and 3 PM and stays for 15 minutes, what is the probability that all three friends will be there at the same time at some point?

p14. Jim the Carpenter starts with a wooden rod of length 1 unit. Jim will cut the middle $\frac{1}{3}$ of the rod and remove it, creating 2 smaller rods of length $\frac{1}{3}$. He repeats this process, randomly choosing a rod to split into 2 smaller rods. Thus, after two such splits, Jim will have 3 rods of length $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{9}$. After 3 splits, Jim will either have 4 rods of lengths $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}, \frac{1}{9}$ or $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}, \frac{1}{27}$. , What is the expected value of the total length of the rods after 5 splits?

p15. Robin is at an archery range. There are 10 targets, each of varying difficulty. If Robin spends t seconds concentrating on target n where $1 \leq n \leq 10$, he has a probability $p = 1 - e^{-t/n}$ of hitting the target. Hitting target n gives him n points and he is alloted a total of 60 seconds. If he has at most one attempt on each target, and he allots time to concentrate on each target optimally to maximize his point total, what is the expected value of the number of points Robin will get? (Assume no time is wasted between shots).

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

–   Guts Round

R1   **p1.** A rectangular pool has diagonal 17 units and area 120 units^2. Joey and Rachel start on opposite sides of the pool when Rachel starts chasing Joey. If Rachel runs 5 units/sec faster than Joey, how long does it take for her to catch him?

p2. Alice plays a game with her standard deck of 52 cards. She gives all of the cards number values where Aces are 1's, royal cards are 10's and all other cards are assigned their face value. Every turn she flips over the top card from her deck and creates a new pile. If the flipped card has value v, she places $12 - v$ cards on top of the flipped card. For example: if she flips the 3 of diamonds then she places 9 cards on top. Alice continues creating piles until she can no longer create a new pile. If the number of leftover cards is 4 and there are 5 piles, what is the sum of the flipped over cards?

p3. There are 5 people standing at $(0,0)$, $(3,0)$, $(0,3)$, $(-3,0)$, and $(-3,0)$ on a coordinate grid at a time $t = 0$ seconds. Each second, every person on the grid moves exactly 1 unit up, down, left, or right. The person at the origin is infected with covid-19, and if someone who is not infected is at the same lattice point as a person who is infected, at any point in time, they will be infected from that point in time onwards. (Note that this means that if two people run into each other at a non-lattice point, such as $(0, 1.5)$, they will not infect each other.) What is the maximum possible number of infected people after $t = 7$ seconds?

p4. Kara gives Kaylie a ring with a circular diamond inscribed in a gold hexagon. The diameter of the diamond is 2 mm. If diamonds cost $\$100/mm^2$ and gold costs $\$50/mm^2$, what is the cost of the ring?

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R2   **p5.** Find the number of three-digit integers that contain at least one 0 or 5. The leading digit of the three-digit integer cannot be zero.

p6. What is the sum of the solutions to $\frac{x+8}{5x+7} = \frac{x+8}{7x+5}$

p7. Let BC be a diameter of a circle with center O and radius 4. Point A is on the circle such that $\angle AOB = 45^o$. Point D is on the circle such that line segmentOD intersects line segment AC at E and OD bisects $\angle AOC$. Compute the area of ADE, which is enclosed by line segments AE and ED and minor arc AD.

p8. William is a bacteria farmer. He would like to give his fiance2021 bacteria as a wedding gift. Since he is an intelligent and frugal bacteria farmer, he would like to add the least amount of

bacteria on his favorite infinite plane petri dish to produce those 2021 bacteria.

The infinite plane petri dish starts off empty and William can add as many bacteria as he wants each day. Each night, all the bacteria reproduce through binary fission, splitting into two. If he has infinite amount of time before his wedding day, how many bacteria should he add to the dish in total to use the least number of bacteria to accomplish his nuptial goals?

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R3   **p9.** The frozen yogurt machine outputs yogurt at a rate of 5 froyo^3/second. If the bowl is described by $z = x^2 + y^2$ and has height 5 froyos, how long does it take to fill the bowl with frozen yogurt?

p10. Prankster Pete and Good Neighbor George visit a street of 2021 houses (each with individual mailboxes) on alternate nights, such that Prankster Pete visits on night 1 and Good Neighbor George visits on night 2, and so on. On each night n that Prankster Pete visits, he drops a packet of glitter in the mailbox of every n^{th} house. On each night m that Good Neighbor George visits, he checks the mailbox of every m^{th} house, and if there is a packet of glitter there, he takes it home and uses it to complete his art project. After the 2021th night, Prankster Pete becomes enraged that none of the houses have yet checked their mail. He then picks three mailboxes at random and takes out a single packet of glitter to dump on George's head, but notices that all of the mailboxes he visited had an odd number of glitter packets before he took one. In how many ways could he have picked these three glitter packets? Assume that each of these three was from a different house, and that he can only visit houses in increasing numerical order.

p11. The taxi-cab length of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $|x_1 - x_2| + |y_1 - y_2|$. Given a series of straight line segments connected head-to-tail, the taxi-cab length of this path is the sum of the taxi-cab lengths of its line segments. A goat is on a rope of taxi-cab length $\frac{7}{2}$ tied to the origin, and it can't enter the house, which is the three unit squares enclosed by $(-2,0),(0,0),(0,-2),(-1,-2),(-1,-1),(-2,-1)$. What is the area of the region the goat can reach? (Note: the rope can't "curve smoothly"-it must bend into several straight line segments.)

p12. Parabola P, $y = ax^2 + c$ has $a > 0$ and $c < 0$. Circle C, which is centered at the origin and lies tangent to P at P's vertex, intersects P at only the vertex. What is the maximum value of a, possibly in terms of c?

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R4   **p13.** Emma has the five letters: A, B, C, D, E. How many ways can she rearrange the letters into

words? Note that the order of words matter, ie $ABCDE$ and $DEABC$ are different.

p14. Seven students are doing a holiday gift exchange. Each student writes their name on a slip of paper and places it into a hat. Then, each student draws a name from the hat to determine who they will buy a gift for. What is the probability that no student draws himself/herself?

p15. We model a fidget spinner as shown below (include diagram) with a series of arcs on circles of radii 1. What is the area swept out by the fidget spinner as it's turned 60° ?
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`png`

p16. Let a, b, c be the sides of a triangle such that $gcd(a, b) = 3528$, $gcd(b, c) = 1008$, $gcd(a, c) = 504$. Find the value of $a * b * c$. Write your answer as a prime factorization.

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R5   **p17.** Let the roots of the polynomial $f(x) = 3x^3 + 2x^2 + x + 8 = 0$ be p, q, and r. What is the sum $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$?

p18. Two students are playing a game. They take a deck of five cards numbered 1 through 5, shuffle them, and then place them in a stack facedown, turning over the top card next to the stack. They then take turns either drawing the card at the top of the stack into their hand, showing the drawn card to the other player, or drawing the card that is faceup, replacing it with the card on the top of the pile. This is repeated until all cards are drawn, and the player with the largest sum for their cards wins. What is the probability that the player who goes second wins, assuming optimal play?

p19. Compute the sum of all primes p such that $2^p + p^2$ is also prime.

p20. In how many ways can one color the 8 vertices of an octagon each red, black, and white, such that no two adjacent sides are the same color?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

R6   **p21**. If $f = \cos(\sin(x))$. Calculate the sum $\sum_{n=0}^{2021} f''(n\pi)$.
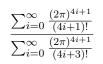
p22. Find all real values of A that minimize the difference between the local maximum and local minimum of $f(x) = \left(3x^2 - 4\right)\left(x - A + \frac{1}{A}\right)$.

p23. Bessie is playing a game. She labels a square with vertices labeled A, B, C, D in clockwise order. There are 7 possible moves: she can rotate her square 90 degrees about the center, 180 degrees about the center, 270 degrees about the center; or she can flip across diagonal AC, flip across diagonal BD, flip the square horizontally (flip the square so that vertices A and B are switched and vertices C and D are switched), or flip the square vertically (vertices B and C are switched, vertices A and D are switched). In how many ways can Bessie arrive back at the original square for the first time in 3 moves?

p24. A positive integer is called *happy* if the sum of its digits equals the two-digit integer formed by its two leftmost digits. Find the number of 5-digit happy integers.

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R7 **p25.** Compute:

$$\frac{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+1)!}}{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+3)!}}$$

p26. Suppose points A, B, C, D lie on a circle ω with radius 4 such that $ABCD$ is a quadrilateral with $AB = 6$, $AC = 8$, $AD = 7$. Let E and F be points on ω such that AE and AF are respectively the angle bisectors of $\angle BAC$ and $\angle DAC$. Compute the area of quadrilateral $AECF$.

p27. Let $P(x) = x^2 - ax + 8$ with a a positive integer, and suppose that P has two distinct real roots r and s. Points $(r, 0)$, $(0, s)$, and (t, t) for some positive integer t are selected on the coordinate plane to form a triangle with an area of 2021. Determine the minimum possible value of $a + t$.

p28. A quartic $p(x)$ has a double root at $x = -\frac{21}{4}$, and $p(x) - 1344x$ has two double roots each $\frac{1}{4}$ less than an integer. What are these two double roots?

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R8 **p29.** Consider pentagon $ABCDE$. How many paths are there from vertex A to vertex E where no edge is repeated and does not go through E.

p30. Let $a_1, a_2, ...$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n = 4$. Compute the maximum possible value of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$ (assume this always converges).

p31. Define function $f(x) = x^4 + 4$. Let

$$P = \prod_{k=1}^{2021} \frac{f(4k-1)}{f(4k-3)}.$$

Find the remainder when P is divided by 1000.

p32. Reduce the following expression to a simplified rational: $\cos^7 \frac{\pi}{9} + \cos^7 \frac{5\pi}{9} + \cos^7 \frac{7\pi}{9}$

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R9   **p33.** Lines ℓ_1 and ℓ_2 have slopes m_1 and m_2 such that $0 < m_2 < m_1$. ℓ_1' and ℓ_2' are the reflections of ℓ_1 and ℓ_2 about the line ℓ_3 defined by $y = x$. Let $A = \ell_1 \cap \ell_2 = (5, 4)$, $B = \ell_1 \cap \ell_3$, $C = \ell_1' \cap \ell_2'$ and $D = \ell_2 \cap \ell_3$. If $\frac{4-5m_1}{-5-4m_1} = m_2$ and $\frac{(1+m_1^2)(1+m_2^2)}{(1-m_1)^2(1-m_2)^2} = 41$, compute the area of quadrilateral $ABCD$.

p34. Suppose $S(m, n) = \sum_{i=1}^{m} (-1)^i i^n$. Compute the remainder when $S(2020, 4)$ is divided by $S(1010, 2)$.

p35. Let N be the number of ways to place the numbers $1, 2, ..., 12$ on a circle such that every pair of adjacent numbers has greatest common divisor 1. What is $N/144$? (Arrangements that can be rotated to yield each other are the same).

p36. Compute the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{2}} = \frac{1}{\binom{2}{2}} - \frac{1}{\binom{4}{2}} + \frac{1}{\binom{6}{2}} - \frac{1}{\binom{8}{2}} - \frac{1}{\binom{10}{2}} + \frac{1}{\binom{12}{2}} + ...$

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

–   Geometry Round

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1   A paper rectangle $ABCD$ has $AB = 8$ and $BC = 6$. After corner B is folded over diagonal AC, what is BD?

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2   Let $ABCD$ be a trapezoid with bases $AB = 50$ and $CD = 125$, and legs $AD = 45$ and $BC = 60$. Find the area of the intersection between the circle centered at B with radius BD and the circle centered at D with radius BD. Express your answer as a common fraction in simplest radical form and in terms of π.

3   If r is a rational number, let $f(r) = \left(\frac{1-r^2}{1+r^2}, \frac{2r}{1+r^2} \right)$. Then the images of f forms a curve in the xy plane. If $f(1/3) = p_1$ and $f(2) = p_2$, what is the distance along the curve between p_1 and p_2?

4 $\triangle \ A_0B_0C_0$ has side lengths $A_0B_0 = 13$, $B_0C_0 = 14$, and $C_0A_0 = 15$. $\triangle \ A_1B_1C_1$ is inscribed in the incircle of $\triangle \ A_0B_0C_0$ such that it is similar to the first triangle. Beginning with $\triangle \ A_1B_1C_1$, the same steps are repeated to construct $\triangle \ A_2B_2C_2$, and so on infinitely many times. What is the value of $\sum_{i=0}^{\infty} A_iB_i$?

5   Let $ABCD$ be a square of side length 1, and let E and F be on the lines AB and AD, respectively, so that B lies between A and E, and D lies between A and F. Suppose that $\angle BCE = 20^o$ and $\angle DCF = 25^o$. Find the area of triangle $\triangle \ EAF$.

6 $\odot A$, centered at point A, has radius 14 and $\odot B$, centered at point B, has radius 15. $AB = 13$. The circles intersect at points C and D. Let E be a point on $\odot A$, and F be the point where line EC intersects $\odot B$, again. Let the midpoints of DE and DF be M and N, respectively. Lines AM and BN intersect at point G. If point E is allowed to move freely on $\odot A$, what is the radius of the locus of G?

7   An n-sided regular polygon with side length 1 is rotated by $\frac{180^o}{n}$ about its center. The intersection points of the original polygon and the rotated polygon are the vertices of a $2n$-sided regular polygon with side length $\frac{1-tan^210^o}{2}$. What is the value of n?

8   In triangle $\triangle \ ABC$, $AB = 5$, $BC = 7$, and $CA = 8$. Let E and F be the feet of the altitudes from B and C, respectively, and let M be the midpoint of BC. The area of triangle MEF can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c such that the greatest common divisor of a and c is 1 and b is not divisible by the square of any prime. Compute $a + b + c$.

9   Rectangle $ABCD$ has an area of 30. Four circles of radius $r_1 = 2$, $r_2 = 3$, $r_3 = 5$, and $r_4 = 4$ are centered on the four vertices A, B, C, and D respectively. Two pairs of external tangents are drawn for the circles at A and C and for the circles at B and D. These four tangents intersect to form a quadrilateral $WXYZ$ where $\overline{WX}$ and $\overline{YZ}$ lie on the tangents through the circles on A and C. If $\overline{WX} + \overline{YZ} = 20$, find the area of quadrilateral $WXYZ$.
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10   In acute $\triangle \ ABC$, let points D, E, and F be the feet of the altitudes of the triangle from A, B,and C, respectively. The area of $\triangle \ AEF$ is 1, the area of $\triangle \ CDE$ is 2, and the area of $\triangle \ BFD$ is

$2 - \sqrt{3}$. What is the area of $\triangle DEF$?

–   Geometry Tiebreaker

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1   What is the radius of the largest circle centered at $(2, 2)$ that is completely bounded within the parabola $y = x^2 - 4x + 5$?

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2   If two points are picked randomly on the perimeter of a square, what is the probability that the distance between those points is less than the side length of the square?

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3   In quadrilateral $ABCD$, $CD = 14$, $\angle BAD = 105^o$, $\angle ACD = 35^o$, and $\angle ACB = 40^o$. Let the midpoint of CD be M. Points P and Q lie on $\overrightarrow{AM}$ and $\overrightarrow{BM}$, respectively, such that $\angle APB = 40^o$ and $\angle AQB = 40^o$. PB intersects CD at point R and QA intersects CD at point S. If $CR = 2$, what is the length of SM?

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