Stanford Mathematics Tournament 2014
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– Team Round

– **p1.** Given that the three points where the parabola $y = bx^2 - 2$ intersects the x-axis and y-axis form an equilateral triangle, compute b.

p2. Compute the last digit of
$$2^{(3^{(4^{\cdots 2014})})}$$

p3. A math tournament has a test which contains 10 questions, each of which come from one of three different subjects. The subject of each question is chosen uniformly at random from the three subjects, and independently of the subjects of all the other questions. The test is unfair if any one subject appears at least 5 times. Compute the probability that the test is unfair.

p4. Let S_n be the sum $S_n = 1 + 11 + 111 + 1111 + ... + 111...11$ where the last number $111...11$ has exactly n 1's. Find $\lfloor 10^{2017}/S_{2014} \rfloor$.

p5. ABC is an equilateral triangle with side length 12. Let O_A be the point inside ABC that is equidistant from B and C and is $\sqrt{3}$ units from A. Define O_B and O_C symmetrically. Find the area of the intersection of triangles O_ABC, AO_BC, and ABO_C.

p6. A *composition* of a natural number n is a way of writing it as a sum of natural numbers, such as $3 = 1 + 2$. Let $P(n)$ denote the sum over all compositions of n of the number of terms in the composition. For example, the compositions of 3 are 3, $1+2$, $2+1$, and $1+1+1$; the first has one term, the second and third have two each, and the last has 3 terms, so $P(3) = 1+2+2+3 = 8$. Compute $P(9)$.

p7. Let ABC be a triangle with $AB = 7$, $AC = 8$, and $BC = 9$. Let the angle bisector of A intersect BC at D. Let E be the foot of the perpendicular from C to line AD. Let M be the midpoint of BC. Find ME.

p8. Call a function g *lower-approximating* for f on the interval $[a, b]$ if for all $x \in [a, b]$, $f(x) \ge g(x)$. Find the maximum possible value of $\int_1^2 g(x)dx$ where $g(x)$ is a linear lower-approximating function for $f(x) = x^x$ on $[1, 2]$.

p9. Determine the smallest positive integer x such that $1.24x$ is the same number as the number obtained by taking the first (leftmost) digit of x and moving it to be the last (rightmost) digit of x.

p10. Let a and b be real numbers chosen uniformly and independently at random from the interval $[-10, 10]$. Find the probability that the polynomial $x^5 + ax + b$ has exactly one real root (ignoring multiplicity).

p11. Let b be a positive real number, and let a_n be the sequence of real numbers defined by $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$ for all $n > 3$. Find the smallest value of b such that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$ diverges.

p12. Find the smallest L such that

$$\left(1-\frac{1}{a}\right)^b\left(1-\frac{1}{2b}\right)^c\left(1-\frac{1}{3c}\right)^a \le L$$

for all real numbers a, b, and c greater than 1.

p13. Find the number of distinct ways in which $30^{(30^{30})}$ can be written in the form $a^{(b^c)}$, where a, b, and c are integers greater than 1.

p14. Convex quadrilateral $ABCD$ has sidelengths $AB = 7$, $BC = 9$, $CD = 15$. A circle with center I lies inside the quadrilateral, and is tangent to all four of its sides. Let M and N be the midpoints of AC and BD, respectively. It can be proven that I always lies on segment MN. If I is in fact the midpoint of MN, find the area of quadrilateral $ABCD$.

p15. Marc has a bag containing 10 balls, each with a different color. He draws out two balls uniformly at random and then paints the first ball he drew to match the color of the second ball. Then he places both balls back in the bag. He repeats until all the balls are the same color. Compute the expected number of times Marc has to perform this procedure before all the balls are the same color.

PS. You had better use hide for answers.

– Geometry Round

1 The coordinates of three vertices of a parallelogram are $A(1, 1)$, $B(2, 4)$, and $C(-5, 1)$. Compute the area of the parallelogram.

2 In a circle, chord AB has length 5 and chord AC has length 7. Arc AC is twice the length of arc AB, and both arcs have degree less than 180. Compute the area of the circle.

3 Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.

4 Let ABC be a triangle such that $AB = 3$, $BC = 4$, and $AC = 5$. Let X be a point in the triangle. Compute the minimal possible value of $AX^2 + BX^2 + CX^2$

5 Let ABC be a triangle where $\angle BAC = 30^\circ$. Construct D in $\triangle ABC$ such that $\angle ABD = \angle ACD = 30^\circ$. Let the circumcircle of $\triangle ABD$ intersect AC at X. Let the circumcircle of $\triangle ACD$ intersect AB at Y. Given that $DB - DC = 10$ and $BC = 20$, find $AX \cdot AY$.

6 Let E be an ellipse with major axis length 4 and minor axis length 2. Inscribe an equilateral triangle ABC in E such that A lies on the minor axis and BC is parallel to the major axis. Compute the area of $\triangle\ ABC$.

7 Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let$D$ and E be the feet of the altitudes from A and B, respectively. Find the circumference of the circumcircle of $\triangle\ CDE$

8 O is a circle with radius 1. A and B are fixed points on the circle such that $AB = \sqrt{2}$. Let C be any point on the circle, and let M and N be the midpoints of AC and BC, respectively. As C travels around circle O, find the area of the locus of points on MN.

9 In cyclic quadrilateral $ABCD$, $AB = AD$. If $AC = 6$ and $\frac{AB}{BD} = \frac{3}{5}$, find the maximum possible area of $ABCD$.

10 Let ABC be a triangle with $AB = 12$, $BC = 5$, $AC = 13$. Let$D$ and E be the feet of the internal and external angle bisectors from B, respectively. (The external angle bisector from B bisects the angle between BC and the extension of AB.) Let ω be the circumcircle of $\triangle\ BDE$, extend AB so that it intersects ω again at F. Extend FC to meet ω again at X, and extend AX to meet ω again at G. Find FG.

– Geometry Tiebreaker

1 A square $ABCD$ with side length 1 is inscribed in a circle. A smaller square lies in the circle with two vertices lying on segment AB and the other two vertices lying on minor arc AB. Compute the area of the smaller square.

2 Let ABC be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let ω be the incircle of ABC with center I. Extend BI so that it intersects AC at E. If ω is tangent to AC at the point D, then find the length of DE.

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3 Circle O has three chords, AD, DF, and EF. Point E lies along the arc AD. Point C is the intersection of chords AD and EF. Point B lies on segment AC such that $EB = EC = 8$. Given $AB = 6$, $BC = 10$, and $CD = 9$, find DF.
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