Rice Math Tournament 2004

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by parmenides51

– Team Round

– **p1.** Find $\sin x - \cos x$ if $\sin 2x = \frac{2002}{2003}$ and $\frac{5\pi}{4} < x < \frac{9\pi}{4}$.

p2. Dave and Daly decide to play a game using two dice. Dave will roll first, then they will continue alternating turns. If Dave gets a total of exactly 6 before Daly gets a total of exactly 7, then Dave wins. Otherwise Daly wins. What is the probability that Dave wins?

p3. Suppose $ABCDEF$ is a regular hexagon with area 1, and consider the diagonals AC, BD, CE, DF, EA and FB. A star is formed by alternating between the vertices of the hexagon and the points of intersection of the diagonals. What is the area of the star?

p4. A pair of positive integers is golden if they end in the same two digits. For example $(139, 2739)$ and $(350, 850)$ are golden pairs. What is the sum of all two-digit integers n for which (n^2, n^3) is golden?

p5. A polynomial is monic if the leading coefficient is 1. Suppose $p(x)$ is a cubic monic polynomial. Let a, b, and c be the roots of $p(x)$. If $a+b+c = 1$ and $a^2+b^2+c^2 = 5$ and $a^3+b^3+c^3 = 16$, determine $p(x)$.

p6. Thirteen students, four of whom are named Bob, are going out to a math conference. They have three distinct cars, two of which will hold 4 students and the other will hold 5. If the students randomly scramble into the cars, what is the probability that every car has at least one Bob in it?

p7. Find all x such that $\sum_{k=1}^{\infty} kx^k = 20$.

p8. Let $\vartriangle ABC$ be an equilateral triangle with side length 2004. For any point P inside $\vartriangle ABC$, let $d(P)$ be the sum of the distances of P to each of the sides of $\vartriangle ABC$. Let m be the minimum value of $d(P)$. Let M be the maximum value of $d(P)$. Compute $M - m$.

p9. There are 30 owls in a row starting from left to right and Sammy (their coach) is facing them. They have a warmup routine involving trading places in rounds. Each round involves three phases. In the first phase of each round, the owls in the odd positions rotate to the right while the owls in even positions do not move. For example, the first owl moves to the third owl's spot who is moving to the fifth owl's spot, etc. Note the 29th owl will wrap around and move to the first owl's spot. The second phase occurs after the first is completed and in this phase the owls in positions which leave a remainder of 1 when divided by 3 rotate to the right. All other owls stay fixed. For example, the owl in position 1 would move to position 4. In the last phase of each round only owls in positions that leave a remainder of 1 when divided by 5 move. These owls however move left! The next round then begins back at phase 1. The warmup continues until the owls are back in their original order. How many rounds does this take?

p10. Let $\lfloor x \rfloor$ denote the floor function (the largest integer less than or equal to x). Find the average value of the quantity $\lfloor 2x^3 - 2\lfloor x^3 \rfloor \rfloor$ on the interval $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

p11. Suppose a king has $25!$ grains of rice after collecting taxes. He is feeling generous so he instructs his accountant to divide the rice into two piles, the first of which he will keep and the second of which he will distribute to the poor. The accountant places the nth grain of rice in the first pile if n and $25!$ are not relatively prime and the second pile otherwise. What fraction of the grains get distributed to the poor?

p12. Given that $x^2 - 3x + 1 = 0$, find $x^9 + x^7 + x^{-9} + x^{-7}$.

p13. How many ordered triples of integers (b, c, d) are there such that $x^4 - 5x^3 + bx^2 + cx + d$ has four (not necessarily distinct) non-negative rational roots?

p14. Find the length of the leading non-repeating block in the decimal expansion of $\frac{2004}{7*5^{2003}}$. (For example, the length of the leading non-repeating block of $\frac{5}{12} = .41666666\overline{6}$ is 2).

p15. Suppose we play a variant of Yahtzee where we roll 5 dice, trying to get all sixes. After each roll we keep the dice that are sixes and re-roll all the others. What is the probability of getting all sixes in exactly k rolls as a function of k?

PS. You had better use hide for answers.

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