Rice Math Tournament 2005

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by parmenides51, Math1331Math

– Team Round

– **p1.** Find the largest prime whose cube divides $1!2!...2005!$.

p2. What is the number of sides of the regular polygon with the largest number of sides whose interior angles measure an integer multiple of 7^o.

p3. A solid is constructed out of an infinite number of cones with height h. The bottom cone has base diameter h. Each successive cone has as its base the circular cross-section halfway up the previous cone. Find the volume of the solid.

p4. Suppose $\triangle ABC$ is a triangle with area 25. Points P, Q and R are on sides AB, BC and CA respectively so that $\frac{BP}{AP} = \frac{CQ}{BQ} = \frac{AR}{CR} = r$. If the area of $\triangle PQR$ is 7, what is the sum of all possible values of r?

p5. You are walking up a staircase with stairs that are 1 ft. tall and 1 ft. deep. The number of stairs from the ground floor to the first floor is 1. From the first floor to the second is 2 stairs. From the 99th to the hundredth stair is 100 steps. At each floor, the staircase makes a 90^o turn to the right. At the top of the 100th floor, how far away from the bottom of the staircase are you?

p6. A point in 3-dimensional space is called a lattice point if all three of its coordinates (x, y, z) are integers. When making a list of lattice points a_1, a_2, ... , a_n, what is the minimum n that guarantees the midpoint between some 2 of the lattice points in the list is a lattice point?

p7. Sparc is played with an octahedral and a dodecahedral die, numbered $1 - 8$ and $1 - 12$. If a player rolls a sum of 2, 6, 11, or 20 he wins. Of the other possible sums, a casino picks some which cause the player to lose. If the player rolls any of the other sums, they roll repeatedly until they get an 11 or their first roll. If they roll an 11 first they lose, if they roll their first roll, they win. Given that the probabilty of winning is $\frac{23242}{110880}$ and that given a choice between two equal probability rolls, the one with greater sum loses, which sums allow the player to keep rolling?

p8. How many right triangles with integer side lengths have one leg (not the hypotenuse) of length 60 ?

p9. Let S be the set of the first nine positive integers, and let A be a nonempty subset of S. The mirror of A is the set formed by replacing each element m of A by $10 - m$. For example, the mirror of $1, 3, 5, 6$ is $4, 5, 7, 9$. A nonempty subset of S is reflective if it is equivalent to its mirror. What is the probability that a randomly chosen nonempty subset of S is reflective?

p10. Approximate to the nearest tenth $\sqrt{2000 \cdot 2010}$.

p11. Each of the small equilateral triangles (9 total) have side length x and is randomly colored red or blue. What is the probability that there will be an equilateral triangle of side length 2x or 3x that is entirely red or entirely blue?
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`png`

p12. Craig Fratrick walks from home to a nearby Dunkin Donuts. He walks East a distance of 30 meters. Then he turns 15^o to the left and walks 30 meters. He repeats this process until he has traveled 210 meters and arrives at Dunkin Donuts. If he had walked directly from home to Dunkin Donuts, how much distance could he have saved by walking directly from home to Dunkin Donuts (in one straight line).

p13. Let

$$P = \cos \frac{\pi}{4} \cos \frac{\pi}{8} ... \cos \frac{\pi}{2^{1000}}.$$

What is $2\pi P$ to the nearest integer?

p14. For all real numbers x, let the mapping $f(x) = \frac{1}{x-i}$. There are real numbers a, b, c and d for which $f(a)$, $f(b)$, $f(c)$, and $f(d)$ form a square in the complex plane. What is the area of the square?

p15. The Fibonacci numbers are defined recursively so that $F_0 = 0$, $F_1 = 1$, and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. Calculate $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \cdot F_n$.

PS. You had better use hide for answers.

– Algebra Round

1 On a test the average score for the girls in class is 91 and the average score for the boys is 85. If the average score for the class is 89, what fraction of the class is boys?

2 How many distinct real roots does the following equation have?

$x^4 + 8x^2 + 16 = 4x^2 - 12x + 9$

3 At William Rice's Marsh there is an intinite number of magic lilypads numbered $1, 2, 3..$ and so on. A magic lilypad lights up if a frog jumps on it while it is not lit, and turns off if a frog jumps on it while it is lit. Suppose all lilypads are initially turned off. Conor the frog begins by hopping on the first lilypad and then hopping on every lilypad thereafter. Bob starts by hopping on the second lilypad and then the fourth, sixth, eight and so on. Dan hops on the third, sixth, ninth lilpyad and so on. If there is a frog for each positive number n that hops on every nth pad, what is the number n the mth lilpyad that remains lit in the end.

4 Ashley Ann Allen, a hapless algebra student sees the expression $\frac{\log A}{\log B}$ and mistakenly reduces the expression to $\frac{A}{B}$ by canceling out the logs. Amazingly when she plugs in the values for A and B she gets the correct answer. Assuming $A \neq B$ find all possible ordered pairs (A, B) such that this is the case

5 Let $f(x)$ and $g(x))$ be functions that take integers as arguments.

Let $f(x + y) = f(x) + g(y) + 8$ for all integers x and y

If $f(x) = x$ for negative integers and let $g(8) = 17$. What is $f(0)$?

6 Let $x = [\frac{2007 \cdot 2006 \cdot 2004 \cdot 2003}{\frac{2005^4}{3}}]$ where $[x]$ is the greatest integer less than or equal to x

Compute $(x^2 + 1)((x^2 + 1) \cdot x^2)^2 + 1)$

7 If the roots of $x^3 + ax^2 + bx + c$ are consecutive integers then what are all possible values of $\frac{a^2}{b+1}$?

8 Find all ordered pairs (a, b) such that the 6 digit number $24ab32$ is divisible by 99

9 Let a, b, c be real numbers such that

$a + b + c = -1$

$a^2 + b^2 + c^2 = 17$

$a^3 + b^3 + c^3 = 11$

Compute abc

10 Find the monic polynomial of least degree that satisfies $p(\sqrt{2} + \sqrt{5}) = 0$