Rice Math Tournament 2012

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– Team Round

– **p1.** How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

p2. Let i be one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Suppose that for all positive integers n, the number n^n never has remainder i upon division by 12. List all possible values of i.

p3. A card is an ordered 4-tuple (a_1, a_2, a_3, a_4) where each a_i is chosen from $\{0, 1, 2\}$. A line is an (unordered) set of three (distinct) cards $\{(a_1, a_2, a_3, a_4)$,(b_1, b_2, b_3, b_4),$(c_1, c_2, c_3, c_4)\}$ such that for each i, the numbers a_i, b_i, c_i are either all the same or all different. How many different lines are there?

p4. We say that the pair of positive integers (x, y), where $x < y$, is a k-tangent pair if we have $\arctan \frac{1}{k} = arctan\frac{1}{x} + arctan\frac{1}{y}$. Compute the second largest integer that appears in a 2012-tangent pair.

p5. Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $i = 1, ..., 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, assuming the convention that $A_{j+6} = A_j$ for all integers j. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

p6. Evaluate $\sum^{\infty}_{n=1} \sum^{\infty}_{m=1} \frac{1}{nm(n+m+1)}$.

p7. A plane in 3-dimensional space passes through the point (a_1, a_2, a_3), with a_1, a_2, and a_3 all positive. The plane also intersects all three coordinate axes with intercepts greater than zero (i.e. there exist positive numbers b_1, b_2, b_3 such that $(b_1, 0, 0)$, $(0, b_2, 0)$, and $(0, 0, b_3)$ all lie on this plane). Find, in terms of a_1, a_2, a_3, the minimum possible volume of the tetrahedron formed by the origin and these three intercepts.

p8. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(ln(t + e))$ centimeters per second, while the child uniformly stretches the rubber band at a rate of one meter per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. If the ant never reaches the right end, answer $+\infty$.

p9. We say that two lattice points are neighboring if the distance between them is 1. We say that a point lies at distance d from a line segment if d is the minimum distance between the point and any point on the line segment. Finally, we say that a lattice point A is nearby a line segment if the distance between A and the line segment is no greater than the distance between the line segment and any neighbor of A. Find the number of lattice points that are nearby the line segment connecting the origin and the point $(1984, 2012)$.

p10. A permutation of the first n positive integers is valid if, for all $i > 1$, i comes after $\lfloor \frac{i}{2} \rfloor$ in the permutation. What is the probability that a random permutation of the first 14 integers is valid?

p11. Given that $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of $\frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}$.

p12. A triangle has sides of length $\sqrt2$, $3+\sqrt3$, and $2\sqrt2+\sqrt6$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

p13. How many positive integers n are there such that for any natural numbers a, b, we have $n|(a^2b + 1)$ implies $n|(a^2 + b)$?

p14. Find constants a and c such that the following limit is finite and nonzero: $c = \lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n-1}{n^a}$. Give your answer in the form (a, c).

p15. Sean thinks packing is hard, so he decides to do math instead. He has a rectangular sheet that he wants to fold so that it fits in a given rectangular box. He is curious to know what the optimal size of a rectangular sheet is so that it's expected to fit well in any given box. Let a and b be positive reals with $a \le b$, and let m and n be independently and uniformly distributed random variables in the interval $(0, a)$. For the ordered 4-tuple (a, b, m, n), let $f(a, b, m, n)$ denote the ratio between the area of a sheet with dimension a×b and the area of the horizontal cross-section of the box with dimension $m \times n$ after the sheet has been folded in halves along each dimension until it occupies the largest possible area that will still fit in the box (because Sean is picky, the sheet must be placed with sides parallel to the box's sides). Compute the smallest value of b/a that maximizes the expectation f.

PS. You had better use hide for answers.

– Geometry Round

1 A circle with radius 1 has diameter AB. C lies on this circle such that ratio of lengths of arcs $AC/BC = 4$. $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

2 In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

3 Let ABC be an equilateral triangle with side length 1. Draw three circles O_a, O_b, and O_c with diameters BC, CA, and AB, respectively. Let S_a denote the area of the region inside O_a and outside of O_b and O_c. Define S_b and S_c similarly, and let S be the area of the region inside all three circles. Find $S_a + S_b + S_c - S$.

4 Let $ABCD$ be a rectangle with area 2012. There exist points E on AB and F on CD such that $DE = EF = FB$. Diagonal AC intersects DE at X and EF at Y . Compute the area of triangle EXY .

5 What is the radius of the largest sphere that fits inside an octahedron of side length 1?

6 A red unit cube $ABCDEFGH$ (with E below A, F below B, etc.) is pushed into the corner of a room with vertex E not visible, so that faces $ABFE$ and $ADHE$ are adjacent to the wall and face $EFGH$ is adjacent to the floor. A string of length 2 is dipped in black paint, and one of its endpoints is attached to vertex A. How much surface area on the three visible faces of the cube can be painted black by sweeping the string over it?

7 Let ABC be a triangle with incircle O and side lengths $5, 8$, and 9. Consider the other tangent line to O parallel to BC, which intersects AB at B_a and AC at C_a. Let r_a be the inradius of triangle AB_aC_a, and define r_b and r_c similarly. Find $r_a + r_b + r_c$.

8 Let ABC be a triangle with side lengths 5, 6, and 7. Choose a radius r and three points outside the triangle O_a, O_b, and O_c, and draw three circles with radius r centered at these three points. If circles O_a and O_b intersect at C, O_b and O_c intersect at A, O_c and O_a intersect at B, and all three circles intersect at a fourth point, find r.

9 In quadrilateral $ABCD$, $m\angle ABD \cong, m\angle BCD$ and $\angle ADB = \angle ABD + \angle BDC$. If $AB = 8$ and $AD = 5$, find BC.

10 A large flat plate of glass is suspended $\sqrt{2/3}$ units above a large flat plate of wood. (The glass is infinitely thin and causes no funny refractive effects.) A point source of light is suspended $\sqrt{6}$ units above the glass plate. An object rests on the glass plate of the following description. Its base is an isosceles trapezoid $ABCD$ with $AB \parallel DC$, $AB = AD = BC = 1$, and $DC = 2$. The

point source of light is directly above the midpoint of CD. The object's upper face is a triangle EFG with $EF = 2$, $EG = FG = \sqrt{3}$. G and AB lie on opposite sides of the rectangle $EFCD$. The other sides of the object are $EA = ED = 1$, $FB = FC = 1$, and $GD = GC = 2$. Compute the area of the shadow that the object casts on the wood plate.