

**Rice Math Tournament 2014**

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– Team Round

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– **p1.** Given that the three points where the parabola  $y = bx^2 - 2$  intersects the  $x$ -axis and  $y$ -axis form an equilateral triangle, compute  $b$ .

**p2.** Compute the last digit of

$$2^{(3^{(4^{\dots 2014}))})}$$

**p3.** A math tournament has a test which contains 10 questions, each of which come from one of three different subjects. The subject of each question is chosen uniformly at random from the three subjects, and independently of the subjects of all the other questions. The test is unfair if any one subject appears at least 5 times. Compute the probability that the test is unfair.

**p4.** Let  $S_n$  be the sum  $S_n = 1 + 11 + 111 + 1111 + \dots + 111\dots 11$  where the last number  $111\dots 11$  has exactly  $n$  1's. Find  $\lfloor 10^{2017} / S_{2014} \rfloor$ .

**p5.**  $ABC$  is an equilateral triangle with side length 12. Let  $O_A$  be the point inside  $ABC$  that is equidistant from  $B$  and  $C$  and is  $\sqrt{3}$  units from  $A$ . Define  $O_B$  and  $O_C$  symmetrically. Find the area of the intersection of triangles  $O_A BC$ ,  $AO_B C$ , and  $ABO_C$ .

**p6.** A *composition* of a natural number  $n$  is a way of writing it as a sum of natural numbers, such as  $3 = 1 + 2$ . Let  $P(n)$  denote the sum over all compositions of  $n$  of the number of terms in the composition. For example, the compositions of 3 are 3, 1 + 2, 2 + 1, and 1 + 1 + 1; the first has one term, the second and third have two each, and the last has 3 terms, so  $P(3) = 1 + 2 + 2 + 3 = 8$ . Compute  $P(9)$ .

**p7.** Let  $ABC$  be a triangle with  $AB = 7$ ,  $AC = 8$ , and  $BC = 9$ . Let the angle bisector of  $A$  intersect  $BC$  at  $D$ . Let  $E$  be the foot of the perpendicular from  $C$  to line  $AD$ . Let  $M$  be the midpoint of  $BC$ . Find  $ME$ .

**p8.** Call a function  $g$  *lower-approximating* for  $f$  on the interval  $[a, b]$  if for all  $x \in [a, b]$ ,  $f(x) \geq g(x)$ . Find the maximum possible value of  $\int_1^2 g(x) dx$  where  $g(x)$  is a linear lower-approximating function for  $f(x) = x^x$  on  $[1, 2]$ .

**p9.** Determine the smallest positive integer  $x$  such that  $1.24x$  is the same number as the number obtained by taking the first (leftmost) digit of  $x$  and moving it to be the last (rightmost) digit of  $x$ .

**p10.** Let  $a$  and  $b$  be real numbers chosen uniformly and independently at random from the interval  $[-10, 10]$ . Find the probability that the polynomial  $x^5 + ax + b$  has exactly one real root (ignoring multiplicity).

**p11.** Let  $b$  be a positive real number, and let  $a_n$  be the sequence of real numbers defined by  $a_1 = a_2 = a_3 = 1$ , and  $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$  for all  $n > 3$ . Find the smallest value of  $b$  such that  $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$  diverges.

**p12.** Find the smallest  $L$  such that

$$\left(1 - \frac{1}{a}\right)^b \left(1 - \frac{1}{2b}\right)^c \left(1 - \frac{1}{3c}\right)^a \leq L$$

for all real numbers  $a, b$ , and  $c$  greater than 1.

**p13.** Find the number of distinct ways in which  $30^{(30^{30})}$  can be written in the form  $a^{(b^c)}$ , where  $a, b$ , and  $c$  are integers greater than 1.

**p14.** Convex quadrilateral  $ABCD$  has sidelengths  $AB = 7$ ,  $BC = 9$ ,  $CD = 15$ . A circle with center  $I$  lies inside the quadrilateral, and is tangent to all four of its sides. Let  $M$  and  $N$  be the midpoints of  $AC$  and  $BD$ , respectively. It can be proven that  $I$  always lies on segment  $MN$ . If  $I$  is in fact the midpoint of  $MN$ , find the area of quadrilateral  $ABCD$ .

**p15.** Marc has a bag containing 10 balls, each with a different color. He draws out two balls uniformly at random and then paints the first ball he drew to match the color of the second ball. Then he places both balls back in the bag. He repeats until all the balls are the same color. Compute the expected number of times Marc has to perform this procedure before all the balls are the same color.

PS. You had better use hide for answers.

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– Geometry Round

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**1** The coordinates of three vertices of a parallelogram are  $A(1, 1)$ ,  $B(2, 4)$ , and  $C(-5, 1)$ . Compute the area of the parallelogram.

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- 2 In a circle, chord  $AB$  has length 5 and chord  $AC$  has length 7. Arc  $AC$  is twice the length of arc  $AB$ , and both arcs have degree less than 180. Compute the area of the circle.
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- 3 Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.
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- 4 Let  $ABC$  be a triangle such that  $AB = 3$ ,  $BC = 4$ , and  $AC = 5$ . Let  $X$  be a point in the triangle. Compute the minimal possible value of  $AX^2 + BX^2 + CX^2$ .
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- 5 Let  $ABC$  be a triangle where  $\angle BAC = 30^\circ$ . Construct  $D$  in  $\triangle ABC$  such that  $\angle ABD = \angle ACD = 30^\circ$ . Let the circumcircle of  $\triangle ABD$  intersect  $AC$  at  $X$ . Let the circumcircle of  $\triangle ACD$  intersect  $AB$  at  $Y$ . Given that  $DB - DC = 10$  and  $BC = 20$ , find  $AX \cdot AY$ .
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- 6 Let  $E$  be an ellipse with major axis length 4 and minor axis length 2. Inscribe an equilateral triangle  $ABC$  in  $E$  such that  $A$  lies on the minor axis and  $BC$  is parallel to the major axis. Compute the area of  $\triangle ABC$ .
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- 7 Let  $ABC$  be a triangle with  $AB = 13$ ,  $BC = 14$ , and  $AC = 15$ . Let  $D$  and  $E$  be the feet of the altitudes from  $A$  and  $B$ , respectively. Find the circumference of the circumcircle of  $\triangle CDE$ .
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- 8  $O$  is a circle with radius 1.  $A$  and  $B$  are fixed points on the circle such that  $AB = \sqrt{2}$ . Let  $C$  be any point on the circle, and let  $M$  and  $N$  be the midpoints of  $AC$  and  $BC$ , respectively. As  $C$  travels around circle  $O$ , find the area of the locus of points on  $MN$ .
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- 9 In cyclic quadrilateral  $ABCD$ ,  $AB = AD$ . If  $AC = 6$  and  $\frac{AB}{BD} = \frac{3}{5}$ , find the maximum possible area of  $ABCD$ .
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- 10 Let  $ABC$  be a triangle with  $AB = 12$ ,  $BC = 5$ ,  $AC = 13$ . Let  $D$  and  $E$  be the feet of the internal and external angle bisectors from  $B$ , respectively. (The external angle bisector from  $B$  bisects the angle between  $BC$  and the extension of  $AB$ .) Let  $\omega$  be the circumcircle of  $\triangle BDE$ , extend  $AB$  so that it intersects  $\omega$  again at  $F$ . Extend  $FC$  to meet  $\omega$  again at  $X$ , and extend  $AX$  to meet  $\omega$  again at  $G$ . Find  $FG$ .
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