Rice Math Tournament 2015

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– Team Round

– **p1.** Two externally tangent unit circles are constructed inside square $ABCD$, one tangent to AB and AD, the other to BC and CD. Compute the length of AB.

p2. We say that a triple of integers (a, b, c) is sorted if $a < b < c$. How many sorted triples of positive integers are there such that $c \le 15$ and the greatest common divisor of a, b, and c is greater than 1?

p3. Two players play a game where they alternate taking a positive integer N and decreasing it by some divisor n of N such that $n < N$. For example, if one player is given $N = 15$, they can choose $n = 3$ and give the other player $N - n = 15 - 3 = 12$. A player loses if they are given $N = 1$.
For how many of the first 2015 positive integers is the player who moves first guaranteed to win, given optimal play from both players?

p4. The polynomial $x^3 - 2015x^2 + mx + n$ has integer coefficients and has 3 distinct positive integer roots. One of the roots is the product of the two other roots. How many possible values are there for n?

p5. You have a robot. Each morning the robot performs one of four actions, each with probability $1/4$: • Nothing. • Self-destruct. • Create one clone. • Create two clones.
Compute the probability that you eventually have no robots.

p6. Four spheres of radius 1 are mutually tangent. What is the radius of the smallest sphere containing them?

p7. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

p8. For some nonzero constant a, let $f(x) = e^{ax}$ and $g(x) = \frac{1}{a}\log x$. Find all possible values of a such that the graphs of f and g are tangent at exactly one point.

p9. Consider a regular pentagon and connect each vertex to the pair of vertices farthest from it by line segments. The line segments intersect at 5 points to form another smaller pentagon. If the large pentagon has side length 1, compute the area of the smaller pentagon. Express your answer without trigonometric functions.

p10. Let f(x) be a function that satisfies $f(x)f(2-x) = x^2f(x-2)$ and $f(1) = \frac{1}{403}$. Compute $f(2015)$.

p11. You are playing a game on the number line. At the beginning of the game, every real number on $[0, 4)$ is uncovered, and the rest are covered. A turn consists of picking a real number r such that, for all x where $r \le x < r + 1$, x is uncovered. The turn ends by covering all such x. At the beginning of a turn, one selects such a real r uniformly at random from among all possible choices for r,; the game ends when no such r exists. Compute the expected number of turns that will take place during this game.

p12. Consider the recurrence $a_{n+1} = 4a_n(1 - a_n)$
Call a point $a_0 \in [0, 1]$ q-*periodic* if $a_q = a_0$. For example, $a_0 = 0$ is always a q-periodic fixed point for any q. Compute the number of positive 2015-periodic fixed points.

p13. Let $a, b, c \in \{-1, 1\}$. Evaluate the following expression, where the sum is taken over all possible choices of a, b, and c:

$$\sum abc\left(2^{\frac{1}{5}} + a2^{\frac{2}{5}} + b2^{\frac{3}{5}} + c2^{\frac{4}{5}}\right)^4.$$

p14. A small circle A of radius $\frac{1}{3}$ rotates, without slipping, inside and tangent to a unit circle B. Let p be a fixed point on A, and compute the length of the closed curve traced out by p as A rotates inside B.

p15. Let x_1, x_2, x_3, x_4, x_5 be distinct positive integers such that $x_1 + x_2 + x_3 + x_4 + x_5 = 100$. Compute the maximum value of the expression

$$\frac{(x_2x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_2-x_1)(x_3-x-1)(x_4-x_1)}+\frac{(x_1x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_1-x2)(x_3-x_2)(x_4-x_2)}+\frac{(x_1x_5+1)(x_2x_5+1)(x_4x_5+1)}{(x_1-x_3)(x_2-x_3)(x_4-x_3)}+\frac{(x_1x_5}{(x_1-}$$

PS. You had better use hide for answers.

– Geometry Round

1 Clyde is making a Pacman sticker to put on his laptop. A Pacman sticker is a circular sticker of radius 3 inches with a sector of 120^o cut out. What is the perimeter of the Pacman sticker in inches?

2 In a certain right triangle, dropping an altitude to the hypotenuse divides the hypotenuse into two segments of length 2 and 3 respectively. What is the area of the triangle?

3 Consider a triangular pyramid $ABCD$ with equilateral base ABC of side length 1. $AD = BD = CD$ and $\angle ADB = \angle BDC = \angle ADC = 90^o$. Find the volume of $ABCD$.

4 Two circles with centers A and B respectively intersect at two points C and D. Given that A, B, C, D lie on a circle of radius 3 and circle A has radius 2, what is the radius of circle B?

5 Consider two concentric circles of radius 1 and 2. Up to rotation, there are two distinct equilateral triangles with two vertices on the circle of radius 2 and the remaining vertex on the circle of radius 1. The larger of these triangles has sides of length a, and the smaller has sides of length b. Compute $a + b$.

6 In a triangle ABC, let D and E trisect BC, so $BD = DE = EC$. Let F be the point on AB such that $\frac{AF}{FB} = 2$, and G on AC such that $\frac{AG}{GC} = \frac{1}{2}$. Let P be the intersection of DG and EF, and extend AP to intersect BC at a point X. Find $\frac{BX}{XC}$

7 A unit sphere is centered at $(0, 0, 1)$. There is a point light source located at $(1, 0, 4)$ that sends out light uniformly in every direction but is blocked by the sphere. What is the area of the sphere's shadow on the $x - y$ plane? (A point (a, b, c) denotes the point in three dimensions with x-coordinate a, y-coordinate b, and z-coordinate c)

8 Consider the parallelogram $ABCD$ such that $CD = 8$ and $BC = 14$. The diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E and $AC = 16$. Consider a point F on the segment $\overline{ED}$ with $FD = \frac{\sqrt{66}}{3}$. Compute CF.

9 Triangle ABC is isoceles with $AB = AC = 2$ and $BC = 1$. Point D lies on AB such that the inradius of ADC equals the inradius of BDC. What is the inradius of ADC?

10 For a positive real number k and an even integer $n \geq 4$, the k-*Perfect* n-gon is defined to be the equiangular n-gon $P_1P_2...P_n$ with $P_iP_{i+1} = P_{n/2+i}P_{n/2+i+1} = k^{i-1}$ for all $i \in \{1, 2, ..., n/2\}$, assuming the convention $P_{n+1} = P_1$ (i.e. the numbering wraps around). If $a(k, n)$ denotes the area of the k-Perfect n-gon, compute $\frac{a(2,24)}{a(4,12)}$.