Rice Math Tournament 2016
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– Team Round

– **p1.** How many squares are there in the xy-plane such that both coordinates of each vertex are integers between 0 and 100 inclusive, and the sides are parallel to the axes?

p2. According to the Constitution of the Kingdom of Nepal, the shape of the flag is constructed as follows:
Draw a line AB of the required length from left to right. From A draw a line AC perpendicular to AB making AC equal to AB plus one third AB. From AC mark off D making line AD equal to line AB. Join BD. From BD mark off E making BE equal to AB. Touching E draw a line FG, starting from the point F on line AC, parallel to AB to the right hand-side. Mark off FG equal to AB. Join CG. If the length of AB is 1 unit, what is the area of the flag?

p3. You have 17 apples and 7 friends, and you want to distribute apples to your friends. The only requirement is that Steven, one of your friends, does not receive more than half of the apples. Given that apples are indistinguishable and friends are distinguishable, compute the number of ways the apples can be distributed.

p4. At $t = 0$ Tal starts walking in a line from the origin. He continues to walk forever at a rate of 1 m/s and lives happily ever after. But Michael (who is Tal's biggest fan) can't bear to say goodbye. At $t = 10$ s he starts running on Tal's path at a rate of n such that $n > 1$ m/s. Michael runs to Tal, gives him a high-five, runs back to the origin, and repeats the process forever. Assuming that the high-fives occur at time $t_0, t_1, t_2, ...,$ compute the limiting value of $\frac{t_z}{t_{z-1}}$ as $z \to \infty$.

p5. In a classroom, there are 47 students in 6 rows and 8 columns. Every student's position is expressed by (i, j). After moving, the position changes to (m, n). Define the change of every student as $(i - m) + (j - n)$. Find the maximum of the sum of changes of all students.

p6. Consider the following family of line segments on the coordinate plane. We take $(0, \pi 2 - a)$ and $(a, 0)$ to be the endpoints of any line segment in the set, for any $0 \le a \le \frac{\pi}{2}$. Let A be the union of all of these line segments. Compute the area of A.

p7. Compute the smallest $n > 2015$ such that $6^n + 8^n$ is divisible by 7.

p8. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

p9. Let C_1 be the circle in the complex plane with radius 1 centered at 0. Let C_2 be the circle in the complex plane with radius 2 centered at $4 - 2i$. Let C_3 be the circle in the complex plane with radius 4 centered at $3 + 8i$. Let S be the set of points which are of the form $\frac{k_1+k_2+k_3}{3}$ where $k_1 \in C_1$, $k_2 \in C_2$, $k_3 \in C_3$. What is the area of S? (Note: a circle or radius r only contains the points at distance r from the center and does not include the points inside the circle)

p10. 3 points are independently chosen at random on a circle. What is the probability that they form an acute triangle?

p11. We say that a number is ascending if its digits are, from left-to-right, in nondecreasing order. We say that a number is descending if the digits are, from left-to-right, in nonincreasing order. Let an be the number of n-digit positive integers which are ascending, and bn be the number of n-digit positive integers which are descending. Compute the ordered pair (x, y) such that $\lim_{n\to\infty} \frac{b_n}{a_n} - x_n - y = 0$.

p12. Let $f(x)$ be a function so that $f(f(x)) = \frac{2x}{1-x^2}$, and $f(x)$ is continuous at all but two points. Compute $f(\sqrt{3})$.

p13. Compute $\sum_{k=1,k\ne m}^{\infty} \frac{1}{(k+m)(k-m)}$

p14. Let $\{x\}$ denote the fractional part of x, the unique real $0 \le \{x\} < 1$ such that $x - \{x\}$ becomes integer. For the function $f_{a,b}(x) = \{x + a\} + 2\{x + b\}$, let its range be $[m_{a,b}, M_{a,b})$. Find the minimum of $M_{a,b}$ as a and b ranges along all reals.

p15. An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is tangent to each of the circles $(x-1)^2+y^2 = 1$ and $(x+1)^2+y^2 = 1$ at two points. Find the ordered pair (a, b) that minimizes the area of the ellipse .

PS. You had better use hide for answers.

– Geometry Round

1 Form a triangle ABC with side lengths $AB = 12$, $AC = 8$, and $BC = 15$. Let the altitude from A to BC intersect BC at D and let AE be the angle bisector of $\angle BAC$, where E is on BC. Compute the length of DE.

2 Consider a unit cube and a plane that slices through it. The plane passes through the midpoints of two adjacent edges on the top face, two on the bottom face, and the center of the cube. Compute the area of the cross section.

3 Compute the area of the largest square that can be inscribed in a unit cube. You may assume that the square's vertces lie on the edges of the cube.

4 Inside a circle of radius 1 are three circles of equal radius such that each of them is tangent to the other two and to the large circle. Determine the radius of one of the smaller circles.

5 When circles of the same radius are packed into the plane with maximum density they form a regular lattice. Compute the packing density of this arrangement, that is, the fraction of area covered by circles.

6 Consider a unit square $ABCD$. Let E be the midpoint of BC and F the intersection of AC and DE. Compute the area of triangle ADF.

7 Consider a circular sector of unit radius and angle $arcsin\left(\frac{1}{3}\right)$. Let S be a square inscribed in the sector such that the axis of symmetry of the sector passes through the center of S, is parallel to two of the sides of S, and all four vertices of S are on the boundary of the sector. What is the area of S?

8 Natasha walks along a closed convex polygonal curve of length 2016. She carries a paintbrush of length 1 and walking all the way around paints all the area as far as she can reach on the outside of the curve. What is that area?

9 Four spheres of radius 1 are mutually tangent. What is the radius of the smallest sphere containing them?

10 Consider a regular pentagon and connect each vertex to the pair of vertices farthest from it by line segments. The line segments intersect at 5 points to form another smaller pentagon. If the large pentagon has side length 1, compute the area of the smaller pentagon. Express your answer without trigonometric functions.