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–	level 3
–	Day 1
1	Let $ABCD$ be a convex quadrilateral in the plane and let O_A, O_B, O_C and O_D be the circumcenters of the triangles BCD, CDA, DAB and ABC, respectively. Suppose these four circumcenters are distinct points. Prove that these points are not on a same circle.
2	Let n be a positive integer. On a $2\times 3n$ board, we mark some squares, so that any square (marked or not) is adjacent to at most two other distinct marked squares (two squares are adjacent when they are distinct and have at least one vertex in common, i.e. they are horizontal, vertical or diagonal neighbors; a square is not adjacent to itself). (a) What is the greatest possible number of marked square? (b) For this maximum number, in how many ways can we mark the squares? configurations that can be achieved through rotation or reflection are considered distinct.
3	Find all positive integers k for which there is an irrational $\alpha > 1$ and a positive integer N such that $\lfloor \alpha^n \rfloor$ is a perfect square minus k for every integer n with $n > N$.
–	Day 2
4	A set A of real numbers is framed when it is bounded and, for all $a, b \in A$, not necessarily distinct, $(a-b)^2 \in A$. What is the smallest real number that belongs to some framed set?
5	Find all triples of non-negative integers (a, b, c) such that $$a^2+b^2+c^2=abc+1.$$
6	Let $n \geq 5$ be integer. The convex polygon $P = A_1A_2\ldots A_n$ is bicentric, that is, it has an inscribed and circumscribed circle. Set $A_{i+n} = A_i$ to every integer i (that is, all indices are taken modulo n). Suppose that for all $i, 1 \leq i \leq n$, the rays $A_{i-1}A_i$ and $A_{i+2}A_{i+1}$ meet at the point B_i. Let ω_i be the circumcircle of $B_iA_iA_{i+1}$. Prove that there is a circle tangent to all n circles ω_i, $1 \leq i \leq n$.
–	junior

1 In a school there are 2021 doors with the numbers $1, 2, \dots, 2021$. In a day 2021 students play the following game: Initially all the doors are closed, and each student receive a card to define the order, there are exactly 2021 cards. The numbers in the cards are $1, 2, \dots, 2020, 2021$.
The order will be student 1 first, student 2 will be the second, and going on. The student k will change the state of the doors $k, 2k, 4k, 8k, \dots, 2^p k$ with $2^p k \le 2021 \le 2^{p+1}k$. Change the state is **if the door was close, it will be open and vice versa.**
a) After the round of the student 16, determine the configuration of the doors $1, 2, \dots, 16$
b) After the round of the student 2021, determine how many doors are closed.

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2 66 points are given on a plane; collinearity is allowed. There are **exactly** 2021 lines passing by at least two of the given points. Determine the greatest number of points in a same line. Give an example.

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3 Let ABC be a scalene triangle and ω is your incircle. The sides BC, CA and AB are tangents to ω in X, Y, Z respectively. Let M be the midpoint of BC and D is the intersection point of BC with the angle bisector of $\angle BAC$. Prove that $\angle BAX = \angle MAC$ if and only if YZ passes by the midpoint of AD.

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4 Let $d(n)$ be the quantity of positive divisors of n, for example $d(1) = 1, d(2) = 2, d(10) = 4$. The **size** of n is k if k is the least positive integer, such that $d^k(n) = 2$. Note that $d^s(n) = d(d^{s-1}(n))$.
a) How many numbers in the interval $[3, 1000]$ have size 2 ?
b) Determine the greatest size of a number in the interval $[3, 1000]$.

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5 Let ABC be an acute-angled triangle. Let A_1 be the midpoint of the arc BC which contain the point A. Let A_2 and A_3 be the foot(s) of the perpendicular(s) of the point A_1 to the lines AB and AC, respectively. Define B_2, B_3, C_2, C_3 analogously.
a) Prove that the line A_2A_3 cuts BC in the midpoint.
b) Prove that the lines A_2A_3, B_2B_3 and C_2C_3 are concurrents.

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6 In a football championship with 2021 teams, each team play with another exactly once. The score of the match(es) is three points to the winner, one point to both players if the match end in draw(tie) and zero point to the loser. The final of the tournament will be played by the two highest score teams. Brazil Football Club won the first match, and it has the advantage if in the final score it draws with any other team. Determine the least score such that Brazil Football Club has a **chance** to play the final match.

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7 Let ABC be a triangle with $\angle ABC = 90^\circ$. The square $BDEF$ is inscribed in $\triangle ABC$, such that D, E, F are in the sides AB, CA, BC respectively. The inradius of $\triangle EFC$ and $\triangle EDA$ are c and b, respectively. Four circles $\omega_1, \omega_2, \omega_3, \omega_4$ are drawn inside the square $BDEF$, such that the radius of ω_1 and ω_3 are both equal to b and the radius of ω_2 and ω_4 are both equal to a. The circle ω_1 is tangent to ED, the circle ω_3 is tangent to BF, ω_2 is tangent to EF and ω_4 is tangent to BD, each one of these circles are tangent to the two closest circles and the circles ω_1 and ω_3 are tangents. Determine the ratio $\frac{c}{a}$.

8 A triple of positive integers (a, b, c) is *brazilian* if

$$a|bc + 1$$

$$b|ac + 1$$

$$c|ab + 1$$

Determine all the brazilian triples.

9 Let $\alpha \geq 1$ be a real number. Define the set

$$A(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots\}$$

Suppose that all the positive integers that **does not belong** to the $A(\alpha)$ are exactly the positive integers that have the same remainder r in the division by 2021 with $0 \leq r < 2021$. Determine all the possible values of α.

