AIME Problems 2022

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–	I
–	February 8, 2022
1	Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.
2	Find the three-digit positive integer $\underline{a}\,\underline{b}\,\underline{c}$ whose representation in base nine is $\underline{b}\,\underline{c}\,\underline{a}_{\text{nine}}$, where a, b, and c are (not necessarily distinct) digits.
3	In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333$. The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.
4	Let $w = \frac{\sqrt{3}+i}{2}$ and $z = \frac{-1+i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.
5	A straight river that is 264 meters wide flows from west to east at a rate of 14 meters per minute. Melanie and Sherry sit on the south bank of the river with Melanie a distance of D meters downstream from Sherry. Relative to the water, Melanie swims at 80 meters per minute, and Sherry swims at 60 meters per minute. At the same time, Melanie and Sherry begin swimming in straight lines to a point on the north bank of the river that is equidistant from their starting positions. The two women arrive at this point simultaneously. Find D.
6	Find the number of ordered pairs of integers (a, b) such that the sequence $$3, 4, 5, a, b, 30, 40, 50$$ is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression.
7	Let $a, b, c, d, e, f, g, h, i$ be distinct integers from 1 to 9. The minimum possible positive value of $$\frac{a \cdot b \cdot c - d \cdot e \cdot f}{g \cdot h \cdot i}$$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8 Equilateral triangle $\triangle ABC$ is inscribed in circle ω with radius 18. Circle ω_A is tangent to sides $\overline{AB}$ and $\overline{AC}$ and is internally tangent to ω. Circles ω_B and ω_C are defined analogously. Circles ω_A, ω_B, and ω_C meet in six points$-$two points for each pair of circles. The three intersection points closest to the vertices of $\triangle ABC$ are the vertices of a large equilateral triangle in the interior of $\triangle ABC$, and the other three intersection points are the vertices of a smaller equilateral triangle in the interior of $\triangle ABC$. The side length of the smaller equilateral triangle can be written as $\sqrt{a} - \sqrt{b}$, where a and b are positive integers. Find $a + b$.

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9 Ellina has twelve blocks, two each of red $(\mathbf{R})$, blue $(\mathbf{B})$, yellow $(\mathbf{Y})$, green $(\mathbf{G})$, orange $(\mathbf{O})$, and purple $(\mathbf{P})$. Call an arrangement of blocks *even* if there is an even number of blocks between each pair of blocks of the same color. For example, the arrangement

R B B Y G G Y R O P P O

is even. Ellina arranges her blocks in a row in random order. The probability that her arrangement is even is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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10 Three spheres with radii 11, 13, and 19 are mutually externally tangent. A plane intersects the spheres in three congruent circles centered at A, B, and C, respectively, and the centers of the spheres all lie on the same side of this plane. Suppose that $AB^2 = 560$. Find AC^2.

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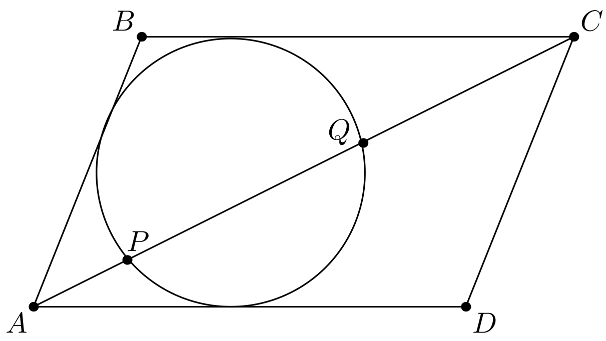
11 Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. A circle tangent to sides $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3$, $PQ = 9$, and $QC = 16$. Then the area of $ABCD$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



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12 For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$

with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

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13 Let S be the set of all rational numbers that can be expressed as a repeating decimal in the form $0.\overline{abcd}$, where at least one of the digits $a, b, c,$ or d is nonzero. Let N be the number of distinct numerators when numbers in S are written as fractions in lowest terms. For example, both 4 and 410 are counted among the distinct numerators for numbers in S because $0.\overline{3636} = \frac{4}{11}$ and $0.\overline{1230} = \frac{410}{3333}$. Find the remainder when N is divided by 1000.

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14 Given $\triangle ABC$ and a point P on one of its sides, call line ℓ the splitting line of $\triangle ABC$ through P if ℓ passes through P and divides $\triangle ABC$ into two polygons of equal perimeter. Let $\triangle ABC$ be a triangle where $BC = 219$ and AB and AC are positive integers. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and suppose that the splitting lines of $\triangle ABC$ through M and N intersect at 30°. Find the perimeter of $\triangle ABC$.

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15 Let x, y, and z be positive real numbers satisfying the system of equations

$$\begin{aligned}\sqrt{2x - xy} + \sqrt{2y - xy} &= 1\\ \sqrt{2y - yz} + \sqrt{2z - yz} &= \sqrt{2}\\ \sqrt{2z - zx} + \sqrt{2x - zx} &= \sqrt{3}.\end{aligned}$$

Then $\left[(1 - x)(1 - y)(1 - z)\right]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

– II

– February 16, 2022

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1 Adults made up $\frac{5}{12}$ of the crowd of people at a concert. After a bus carrying 50 more people arrived, adults made up $\frac{11}{25}$ of the people at the concert. Find the minimum number of adults who could have been at the concert after the bus arrived.

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2 Azar, Carl, Jon, and Sergey are the four players left in a singles tennis tournament. They are randomly assigned opponents in the semifinal matches, and the winners of those matches play each other in the final match to determine the winner of the tournament. When Azar plays Carl, Azar will win the match with probability $\frac{2}{3}$. When either Azar or Carl plays either Jon or Sergey, Azar or Carl will win the match with probability $\frac{3}{4}$. Assume that outcomes of different matches are independent. The probability that Carl will win the tournament is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

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3 A right square pyramid with volume 54 has a base with side length 6. The five vertices of the pyramid all lie on a sphere with radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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4 There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}(\frac{m}{n})$, where m and n are relatively prime positive integers. Find $m+n$.

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5 Twenty distinct points are marked on a circle and labeled 1 through 20 in clockwise order. A line segment is drawn between every pair of points whose labels differ by a prime number. Find the number of triangles formed whose vertices are among the original 20 points.

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6 Let $x_1 \leq x_2 \leq \cdots \leq x_{100}$ be real numbers such that $|x_1| + |x_2| + \cdots + |x_{100}| = 1$ and $x_1 + x_2 + \cdots + x_{100} = 0$. Among all such 100-tuples of numbers, the greatest value that $x_{76} - x_{16}$ can achieve is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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7 A circle with radius 6 is externally tangent to a circle with radius 24. Find the area of the triangular region bounded by the three common tangent lines of these two circles.

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8 Find the number of positive integers $n \leq 600$ whose value can be uniquely determined when the values of $\left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{5} \right\rfloor$, and $\left\lfloor \frac{n}{6} \right\rfloor$ are given, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x.

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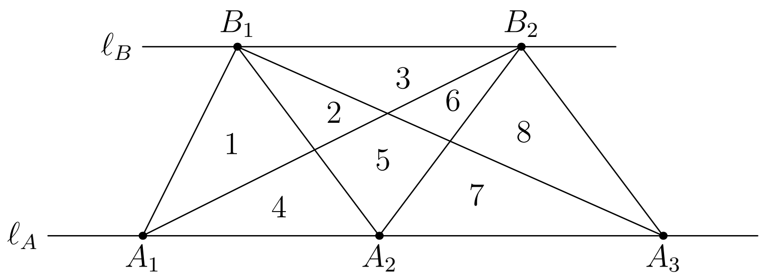
9 Let ℓ_A and ℓ_B be two distinct parallel lines. For positive integers m and n, distinct points $A_1, A_2, A_3, \ldots, A_m$ lie on ℓ_A, and distinct points $B_1, B_2, B_3, \ldots, B_n$ lie on ℓ_B. Additionally, when segments $\overline{A_iB_j}$ are drawn for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$, no point strictly between ℓ_A and ℓ_B lies on more than two of the segments. Find the number of bounded regions into which this figure divides the plane when $m = 7$ and $n = 5$. The figure shows that there are 8 regions when $m = 3$ and $n = 2$.



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10 Find the remainder when

$$\binom{\binom{3}{2}}{2}+\binom{\binom{4}{2}}{2}+\cdots+\binom{\binom{40}{2}}{2}$$

is divided by 1000.

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11 Let $ABCD$ be a convex quadrilateral with $AB=2$, $AD=7$, and $CD=3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of $\overline{BC}$. Find the square of the area of $ABCD$.

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12 Let $a, b, x,$ and y be real numbers with $a>4$ and $b>1$ such that

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-16}=\frac{(x-20)^2}{b^2-1}+\frac{(y-11)^2}{b^2}=1.$$

Find the least possible value of $a+b$.

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13 There is a polynomial $P(x)$ with integer coefficients such that

$$P(x)=\frac{(x^{2310}-1)^6}{(x^{105}-1)(x^{70}-1)(x^{42}-1)(x^{30}-1)}$$

holds for every $0<x<1$. Find the coefficient of x^{2022} in $P(x)$

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14 For positive integers a, b, and c with $a<b<c$, consider collections of postage stamps in denominations a, b, and c cents that contain at least one stamp of each denomination. If there exists such a collection that contains sub-collections worth every whole number of cents up to 1000 cents, let $f(a,b,c)$ be the minimum number of stamps in such a collection. Find the sum of the three least values of c such that $f(a,b,c)=97$ for some choice of a and b.

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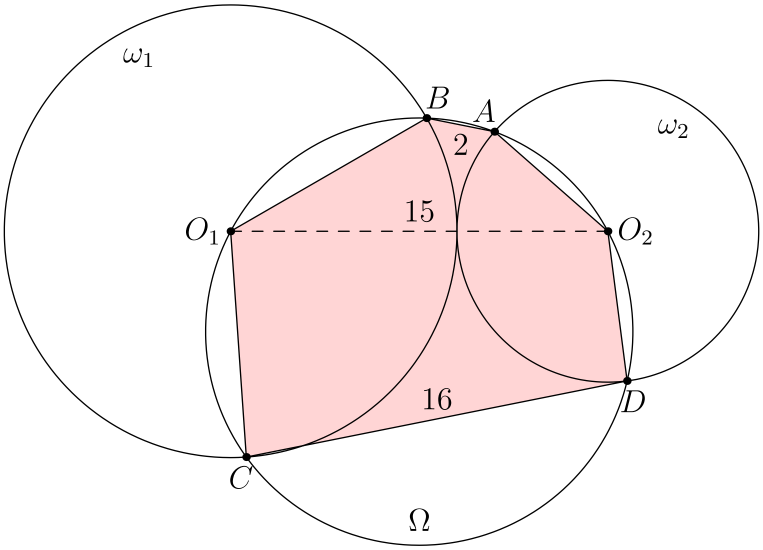
15 Two externally tangent circles ω_1 and ω_2 have centers O_1 and O_2, respectively. A third circle Ω passing through O_1 and O_2 intersects ω_1 at B and C and ω_2 at A and D, as shown. Suppose that $AB=2$, $O_1O_2=15$, $CD=16$, and ABO_1CDO_2 is a convex hexagon. Find the area of this hexagon.


ω_1
B
A
ω_2
2
15
O_1
O_2
16
D
C
Ω