EGMO TST - Romania 2022

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by oVlad, April

–	Day 1
P1	A finite set M of real numbers has the following properties: M has at least 4 elements, and there exists a bijective function $f : M \to M$, different from the identity, such that $ab \leq f(a)f(b)$ for all $a \neq b \in M$. Prove that the sum of the elements of M is 0.
P2	At first, on a board, the number 1 is written 100 times. Every minute, we pick a number a from the board, erase it, and write $a/3$ thrice instead. We say that a positive integer n is *persistent* if after any amount of time, regardless of the numbers we pick, we can find at least n equal numbers on the board. Find the greatest persistent number.
P3	Let $ABCD$ be a convex quadrilateral and let O be the intersection of its diagonals. Let $P, Q, R,$ and S be the projections of O on $AB, BC, CD,$ and DA respectively. Prove that $$2(OP+OQ+OR+OS) \leq AB+BC+CD+DA.$$
P4	For every positive integer $N \geq 2$ with prime factorisation $N = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ we define $$f(N) := 1 + p_1a_1 + p_2a_2 + \cdots + p_ka_k.$$ Let $x_0 \geq 2$ be a positive integer. We define the sequence $x_{n+1} = f(x_n)$ for all $n \geq 0$. Prove that this sequence is eventually periodic and determine its fundamental period.
–	Day 2
P1	Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that all real numbers x and y satisfy $$f(f(x)+y) = f(x^2-y) + 4f(x)y.$$
P2	On a board there is a regular polygon $A_1A_2\ldots A_{99}$. Ana and Barbu alternatively occupy empty vertices of the polygon and write down triangles on a list: Ana only writes obtuse triangles, while Barbu only writes acute ones. At the first turn, Ana chooses three vertices X, Y and Z and writes down $\triangle XYZ$. Then, Barbu chooses two of X, Y and Z, for example X and Y, and an unchosen vertex T, and writes down $\triangle XYT$. The game goes on and at each turn, the player must choose a new vertex R and write

down $\triangle PQR$, where P is the last vertex chosen by the other player, and Q is one of the other vertices of the last triangle written down by the other player.

If one player cannot perform a move, then the other one wins. If both people play optimally, determine who has a winning strategy.

P3 Let be given a parallelogram $ABCD$ and two points A_1, C_1 on its sides AB, BC, respectively. Lines AC_1 and CA_1 meet at P. Assume that the circumcircles of triangles AA_1P and CC_1P intersect at the second point Q inside triangle ACD. Prove that $\angle PDA = \angle QBA$.

P4 Let $p \geq 3$ be an odd positive integer. Show that p is prime if and only if however we choose $(p+1)/2$ pairwise distinct positive integers, we can find two of them, a and b, such that $(a+b)/\gcd(a,b) \geq p$.
