Latvia National Olympiad 2016

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–	Third Round, Grade 9
1	Given positive integers x and y such that xy^2 is a perfect cube, prove that x^2y is also a perfect cube.
2	Triangle ABC has median AF, and D is the midpoint of the median. Line CD intersects AB in E. Prove that $BD = BF$ implies $AE = DE$!
3	Is it possible to insert numbers $1, \dots, 16$ into a table 4×4 (each cell should have a different number) so that every two adjacent cells (i.e. cells sharing a common side) have numbers a and b satisfying (a) $\|a-b\| \geq 6$ (b) $\|a-b\| \geq 7$
4	Find the least prime factor of the number $\frac{2016^{2016}-3}{3}$.
5	The integer sequence (s_i) "having pattern 2016"' is defined as follows: ○ The first member s_1 is 2. ○ The second member s_2 is the least positive integer exceeding s_1 and having digit 0 in its decimal notation. ○ The third member s_3 is the least positive integer exceeding s_2 and having digit 1 in its decimal notation. ○ The third member s_3 is the least positive integer exceeding s_2 and having digit 6 in its decimal notation. The following members are defined in the same way. The required digits change periodically: $2 \to 0 \to 1 \to 6 \to 2 \to 0 \to \dots$. The first members of this sequence are the following: $2; 10; 11; 16; 20; 30; 31; 36; 42; 50$. What are the 4 numbers that immediately follow $s_k = 2016$ in this sequence?
–	Third Round, Grade 10
1	Given that x and y are positive integers such that xy^{10} is perfect 33rd power of a positive integer, prove that $x^{10}y$ is also a perfect 33rd power!

2 The bisectors of the angles $\sphericalangle CAB$ and $\sphericalangle BCA$ intersect the circumcircle of ABC in P and Q respectively. These bisectors intersect each other in point I. Prove that $PQ \perp BI$.

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3 Assume that real numbers x, y and z satisfy $x + y + z = 3$. Prove that $xy + xz + yz \leq 3$.

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4 In a Pythagorean triangle all sides are longer than 5. Is it possible that (a) all three sides are prime numbers, (b) exactly two sides are prime numbers. (Note: We call a triangle "Pythagorean", if it is a right-angled triangle where all sides are positive integers.)

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5 All vertices of a regular 2016-gon are initially white. What is the least number of them that can be painted black so that:

(a) There is no right triangle

(b) There is no acute triangle

having all vertices in the vertices of the 2016-gon that are still white?

– Third Round, Grade 11

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1 Given that x and y are positive integers such that xy^{433} is a perfect 2016-power of a positive integer, prove that $x^{433}y$ is also a perfect 2016-power.

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2 An acute triangle ABC ($AB > AC$) has circumcenter O, but D is the midpoint of BC. Circle with diameter AD intersects sides AB and AC in E and F respectively. On segment EF pick a point M so that $DM \parallel AO$. Prove that triangles ABD and FDM are similar.

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3 Prove that for every integer n ($n > 1$) there exist two positive integers x and y ($x \leq y$) such that

$$\frac{1}{n} = \frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \cdots + \frac{1}{y(y+1)}$$

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4 The integer sequence (s_i) "having pattern 2016"' is defined as follows: ∘ The first member s_1 is 2. ∘ The second member s_2 is the least positive integer exceeding s_1 and having digit 0 in its decimal notation. ∘ The third member s_3 is the least positive integer exceeding s_2 and having digit 1 in its decimal notation. ∘ The third member s_3 is the least positive integer exceeding s_2 and having digit 6 in its decimal notation.
The following members are defined in the same way. The required digits change periodically: $2 \to 0 \to 1 \to 6 \to 2 \to 0 \to \ldots$. The first members of this sequence are the following: $2; 10; 11; 16; 20; 30; 31; 36; 42; 50$.

Does this sequence contain a) 2001, b) 2006?

5 Prove that every triangle can be cut into three pieces so that every piece has axis of symmetry. Show how to cut it (a) using three line segments, (b) using two line segments.

– Third Round, Grade 12

1 Given that x, y and z are positive integers such that $x^3y^5z^6$ is a perfect 7th power of a positive integer, show that also $x^5y^6z^3$ is a perfect 7th power.

2 Triangle ABC has incircle ω and incenter I. On its sides AB and BC we pick points P and Q respectively, so that $PI = QI$ and $PB > QB$. Line segment QI intersects ω in T. Draw a tangent line to ω passing through T; it intersects the sides AB and BC in U and V respectively. Prove that $PU = UV + VQ$!

3 Prove that among any 18 consecutive positive 3-digit numbers, there is at least one that is divisible by the sum of its digits!

4 Two functions are defined by equations: $f(a) = a^2 + 3a + 2$ and $g(b, c) = b^2 - b + 3c^2 + 3c$. Prove that for any positive integer a there exist positive integers b and c such that $f(a) = g(b, c)$.

5 Consider the graphs of all the functions $y = x^2 + px + q$ having 3 different intersection points with the coordinate axes. For every such graph we pick these 3 intersection points and draw a circumcircle through them. Prove that all these circles have a common point!