Iran TST 2016

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Test 1 Day 1

1 Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

2 For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

3 Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Test 1 Day 2

4 Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

5 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

6 In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

Proposed by Russia

Test 2 Day 1

1 Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

2 Let a, b, c, d be positive real numbers such that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 2$. Prove that

$$\sum_{cyc} \sqrt{\frac{a^2+1}{2}} \geq (3.\sum_{cyc} \sqrt{a}) - 8$$

3 Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Proposed by Finland

Test 2 Day 2

4 Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Proposed by El Salvador

5 Let P and P' be two unequal regular $n-$gons and A and A'two points inside P andP', respectively.Suppose $\{d_1, d_2, \cdots d_n\}$ are the distances from A to the vertices of P and $\{d'_1, d'_2, \cdots d'_n\}$ are defines similarly for P', A'. Is it possible for $\{d'_1, d'_2, \cdots d'_n\}$ to be a permutation of $\{d_1, d_2, \cdots d_n\}$?

6 Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f: \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called [i]k-good[/i] if $\gcd(f(m)+n, f(n)+m) \leq k$ for all $m \neq n$. Find all k such that there exists a k-good function.

Proposed by James Rickards, Canada

Test 3 Day 1

1 A real function has been assigned to every cell of an $n \times n$ table. Prove that a function can be assigned to each row and each column of this table such that the function assigned to each cell is equivalent to the combination of functions assigned to the row and the column containing it.

2 Let ABC be an arbitrary triangle and O is the circumcenter of $\triangle ABC$.Points X,Y lie on AB,AC,respectively such that the reflection of BC WRT XY is tangent to circumcircle of $\triangle AXY$.Prove that the circumcircle of triangle AXY is tangent to circumcircle of triangle BOC.

3 Let $p \neq 13$ be a prime number of the form $8k+5$ such that 39 is a quadratic non-residue modulo p. Prove that the equation

$$x_1^4+x_2^4+x_3^4+x_4^4 \equiv 0 \pmod p$$

has a solution in integers such that $p \nmid x_1x_2x_3x_4$.

Test 3 Day 2

4 Suppose that a sequence $a_1,a_2,\ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2+(k-1)}$$

for every positive integer k. Prove that $a_1+a_2+\ldots+a_n \geq n$ for every $n \geq 2$.

5 Let AD,BF,CE be altitudes of triangle ABC.Q is a point on EF such that $QF=DE$ and F is between E,Q.P is a point on EF such that $EP=DF$ and E is between P,F.Perpendicular bisector of DQ intersect with AB at X and perpendicular bisector of DP intersect with AC at Y.Prove that midpoint of BC lies on XY.

6 Suppose that a council consists of five members and that decisions in this council are made according to a method based on the positive or negative vote of its members. The method used by this council has the following two properties:

• **Ascension:**If the presumptive final decision is favorable and one of the opposing members changes his/her vote, the final decision will still be favorable. • **Symmetry:** If all of the members change their vote, the final decision will change too.

Prove that the council uses a weighted decision-making method ; that is , nonnegative weights $\omega_1,\omega_2,\cdots,\omega_5$ can be assigned to members of the council such that the final decision is favorable if and only if sum of the weights of those in favor is greater than sum of the weights of the rest.

Remark. The statement isn't true at all if you replace 5 with arbitrary n . In fact , finding a counter example for $n = 6$, was appeared in the same year's Iran MO 2nd round P6 (https://artofproblemsolving.com/community/c6h1459567p8417532)
