Math Majors of America Tournament for High Schools 2014

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by parmenides51

– Mathathon Round

– Round 1

p1. A circle is inscribed inside a square such that the cube of the radius of the circle is numerically equal to the perimeter of the square. What is the area of the circle?

p2. If the coefficient of z^ky^k is 252 in the expression $(z + y)^{2k}$, find k.

p3. Let $f(x) = \frac{4x^4-2x^3-x^2-3x-2}{x^4-x^3+x^2-x-1}$ be a function defined on the real numbers where the denominator is not zero. The graph of f has a horizontal asymptote. Compute the sum of the x-coordinates of the points where the graph of f intersects this horizontal asymptote. If the graph of f does not intersect the asymptote, write 0.

Round 2

p4. How many 5-digit numbers have strictly increasing digits? For example, 23789 has strictly increasing digits, but 23889 and 23869 do not.

p5. Let

$$y = \frac{1}{1 + \frac{1}{9+\frac{1}{5+\frac{1}{9+\frac{1}{5+...}}}}}$$

If y can be represented as $\frac{a\sqrt{b}+c}{d}$, where b is not divisible by any squares, and the greatest common divisor of a and d is 1, find the sum $a + b + c + d$.

p6. "Counting" is defined as listing positive integers, each one greater than the previous, up to (and including) an integer n. In terms of n, write the number of ways to count to n.

Round 3

p7. Suppose p, q, $2p^2 + q^2$, and $p^2 + q^2$ are all prime numbers. Find the sum of all possible values of p.

p8. Let $r(d)$ be a function that reverses the digits of the 2-digit integer d. What is the smallest 2-digit positive integer N such that for some 2-digit positive integer n and 2-digit positive integer $r(n)$, N is divisible by n and $r(n)$, but not by 11?

p9. What is the period of the function $y = (\sin(3\theta) + 6)^2 - 10(sin(3\theta) + 7) + 13$?

Round 4

p10. Three numbers a, b, c are given by $a = 2^2(\sum_{i=0}^{2} 2^i)$, $b = 2^4(\sum_{i=0}^{4} 2^i)$, and $c = 2^6(\sum_{i=0}^{6} 2^i)$. u, v, w are the sum of the divisors of a, b, c respectively, yet excluding the original number itself. What is the value of $a + b + c - u - v - w$?

p11. Compute $\sqrt{6 - \sqrt{11}} - \sqrt{6 + \sqrt{11}}$.

p12. Let $a_0, a_1, ..., a_n$ be such that $a_n \neq 0$ and

$$(1 + x + x^3)^{341}(1 + 2x + x^2 + 2x^3 + 2x^4 + x^6)^{342} = \sum_{i=0}^{n} a_i x^i.$$

Find the number of odd numbers in the sequence $a_0, a_1, ..., a_n$.

PS. You should use hide for answers. Rounds 5-7 have been posted here (https://artofproblemsolving.com/community/c4h2781343p24424617). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 5

p13. How many ways can we form a group with an odd number of members (plural) from 99 people? Express your answer in the form $a^b + c$, where a, b, and c are integers and a is prime.

p14. A cube is inscibed in a right circular cone such that the ratio of the height of the cone to the radius is $2 : 1$. Compute the fraction of the cone's volume that the cube occupies.

p15. Let $F_0 = 1$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

Round 6

p16. Ankit finds a quite peculiar deck of cards in that each card has n distinct symbols on it and any two cards chosen from the deck will have exactly one symbol in common. The cards are guaranteed to not have a certain symbol which is held in common with all the cards. Ankit decides to create a function f(n) which describes the maximum possible number of cards in a set given the previous constraints. What is the value of $f(10)$?

p17. If $|x| < \frac{1}{4}$ and
$$X = \sum_{N=0}^{\infty}\sum_{n=0}^{N}\binom{N}{n}x^{2n}(2x)^{N-n}.$$
then write X in terms of x without any summation or product symbols (and without an infinite number of '$+$'s, etc.).

p18. Dietrich is playing a game where he is given three numbers a, b, c which range from $[0, 3]$ in a continuous uniform distribution. Dietrich wins the game if the maximum distance between any two numbers is no more than 1. What is the probability Dietrich wins the game?

Round 7

p19. Consider f defined by
$$f(x) = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6.$$
How many tuples of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6)$ exist such that $f(-1) = 12$ and $f(1) = 30$?

p20. Let a_n be the number of permutations of the numbers $S = \{1, 2, ..., n\}$ such that for all k with $1 \le k \le n$, the sum of k and the number in the kth position of the permutation is a power of 2. Compute $a_1 + a_2 + a_4 + a_8 + ... + a_{1048576}$.

p21. A 4-dimensional hypercube of edge length 1 is constructed in 4-space with its edges parallel to the coordinate axes and one vertex at the origin. Its coordinates are given by all possible permutations of $(0, 0, 0, 0)$,$(1, 0, 0, 0)$,$(1, 1, 0, 0)$,$(1, 1, 1, 0)$, and $(1, 1, 1, 1)$. The 3-dimensional hyperplane given by $x + y + z + w = 2$ intersects the hypercube at 6 of its vertices. Compute the 3-dimensional volume of the solid formed by the intersection.

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c4h2781335p24424563). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Round

– **p1.** For what value of $x > 0$ does $f(x) = (x-3)^2(x+4)$ achieve the smallest value?

p2. There are exactly 29 possible values that can be made using one or more of the 5 distinct coins with values 1, 3, 5, 7, and X. What is the smallest positive integral value for X?

p3. Define $\star$ as $x \star y = x - \frac{1}{xy}$. What is the sum of all complex x such that $x \star (x \star 2x) = 2x$?

p4. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and let $\lceil x \rceil$ be the least integer greater than or equal to x. Compute the smallest positive value of a for which $\lfloor a \rfloor$, $\lceil a \rceil$, $\lfloor a^2 \rfloor$ is a non-constant arithmetic sequence.

p5. A right triangle is bounded in a coordinate plane by the lines $x = 0$, $y = 0$, $x = x_{100}$, and $y = f(x)$, where f is a linear function with a negative slope and $f(x_{100}) = 0$. The lines $x = x_1$, $x = x_2$, ... , $x = x_{99}$ ($x_1 < x_2 < ... < x_{100}$) subdivide the triangle into 100 regions of equal area. Compute $\frac{x_{100}}{x_1}$.

p6. There are 10 children in a line to get candy. The pieces of candy are indistinguishable, while the children are not. If there are a total of 390 pieces of candy, how many ways are there to distribute the candy so that the n^{th} child in line receives at least n^2 pieces of candy?

p7. Compute

$$\frac{\binom{54}{23} + 6\binom{54}{24} + 15\binom{54}{25} + 15\binom{54}{27} + 6\binom{54}{28} + \binom{54}{29} - \binom{60}{29}}{\binom{54}{26}}$$

p8. Point A lies on the circle centered at O. $\overline{AB}$ is tangent to O, and C is located on the circle so that $m\angle AOC = 120^o$ and oriented so that $\angle BAC$ is obtuse. $\overline{BC}$ intersects the circle at D. If $AB = 6$ and $BD = 3$, then compute the radius of the circle.

p9. The center of each face of a regular octahedron (a solid figure with 8 equilateral triangles as faces) with side length one unit is marked, and those points are the vertices of some cube. The center of each face of the cube is marked, and these points are the vertices of an even smaller regular octahedron. What is the volume of the smaller octahedron?

p10. Compute the greatest positive integer n such that there exists an odd integer a, for which $\frac{a^{2^n}-1}{4^{4^4}}$ is not an integer.

p11. Three identical balls are painted white and black, so that half of each sphere is a white hemisphere, and the other half is a black one. The three balls are placed on a plane surface, each with a random orientation, so that each ball has a point of contact with the other two. What is the probability that at at least one point of contact between two of the balls, both balls are the same color?

p12. Define an operation Φ whose input is a real-valued function and output is a real number so that it has the following properties: • For any two real-valued functions $f(x)$ and $g(x)$, and any real numbers a and b, then

$$\Phi(af(x) + bg(x)) = a\Phi(f(x)) + b\Phi(g(x))$$

• For any real-valued function $h(x)$, there is a polynomial function $p(x)$ such that

$$\Phi(p(x) \cdot h(x)) = \Phi((h(x))^2)$$

• If some function $m(x)$ is always non-negative, and $\Phi(m(x)) = 0$, then $m(x)$ is always 0.
Let $r(x)$ be a real-valued function with $r(5) = 3$. Let S be the set of all real-valued functions $s(x)$ that satisfy that $\Phi(r(x) \cdot x^n) = \Phi(s(x) \cdot x^{n+1})$. For each s in S, give the value of $s(5)$.

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–	Tiebreaker Round
1	Show that there does not exist a right triangle with all integer side lengths such that exactly one of the side lengths is odd.
2	Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers with $a_1 < a_2 < a_3 < ...$, and for n = 1, 2, 3,..., $$a_{2n} = a_n + n.$$ Furthermore, whenever n is prime, so is a_n. Prove that $a_n = n$.
3	Let $f : R^+ \to R^+$ be a function satisfying $$f(\sqrt{x_1x_2}) = \sqrt{f(x_1)f(x_2)}$$ for all positive real numbers x_1, x_2. Show that $$f(\sqrt[n]{x_1x_2...x_n}) = \sqrt[n]{f(x_1)f(x_2)...f(x_n)}$$ for all positive integers n and positive real numbers $x_1, x_2, ..., x_n$.

4 Determine, with proof, the maximum and minimum among the numbers

$$\sqrt{5}-\lfloor\sqrt{5}\rfloor, 2\sqrt{5}-\lfloor 2\sqrt{5}\rfloor, 3\sqrt{5}-\lfloor 3\sqrt{5}\rfloor, ..., 2013\sqrt{5}-\lfloor 2013\sqrt{5}\rfloor, 2014\sqrt{5}-\lfloor 2014\sqrt{5}\rfloor$$

– Mixer Round

Mixer Round **p1.** How many real roots does the equation $2x^7 + x^5 + 4x^3 + x + 2 = 0$ have?

p2. Given that $f(n) = 1 + \sum_{j=1}^{n}(1 + \sum_{i=1}^{j}(2i + 1))$, find the value of $f(99) - \sum_{i=1}^{99} i^2$.

p3. A rectangular prism with dimensions $1 \times a \times b$, where $1 < a < b < 2$, is bisected by a plane bisecting the longest edges of the prism. One of the smaller prisms is bisected in the same way. If all three resulting prisms are similar to each other and to the original box, compute ab. Note: Two rectangular prisms of dimensions $p \times q \times r$ and$x \times y \times z$ are similar if $\frac{p}{x} = \frac{q}{y} = \frac{r}{z}$.

p4. For fixed real values of p, q, r and s, the polynomial $x^4 + px^3 + qx^2 + rx + s$ has four non real roots. The sum of two of these roots is $4 + 7i$, and the product of the other two roots is $3 - 4i$. Compute q.

p5. There are 10 seats in a row in a theater. Say we have an infinite supply of indistinguishable good kids and bad kids. How many ways can we seat 10 kids such that no two bad kids are allowed to sit next to each other?

p6. There are an infinite number of people playing a game. They each pick a different positive integer k, and they each win the amount they chose with probability $\frac{1}{k^3}$. What is the expected amount that all of the people win in total?

p7. There are 100 donuts to be split among 4 teams. Your team gets to propose a solution about how the donuts are divided amongst the teams. (Donuts may not be split.) After seeing the proposition, every team either votes in favor or against the propisition. The proposition is adopted with a majority vote or a tie. If the proposition is rejected, your team is eliminated and will never receive any donuts. Another remaining team is chosen at random to make a proposition, and the process is repeated until a proposition is adopted, or only one team is left. No promises or deals need to be kept among teams besides official propositions and votes. Given that all teams play optimally to maximize the expected value of the number of donuts they receive, are completely indifferent as to the success of the other teams, but they would rather not eliminate a team than eliminate one (if the number of donuts they receive is the same either way), then how much should your team propose to keep?

p8. Dominic, Mitchell, and Sitharthan are having an argument. Each of them is either credible or not credible – if they are credible then they are telling the truth. Otherwise, it is not known whether they are telling the truth. At least one of Dominic, Mitchell, and Sitharthan is credible. Tim knows whether Dominic is credible, and Ethan knows whether Sitharthan is credible. The following conversation occurs, and Tim and Ethan overhear:
Dominic: "Sitharthan is not credible."
Mitchell: "Dominic is not credible."
Sitharthan: "At least one of Dominic or Mitchell is credible."
Then, at the same time, Tim and Ethan both simultaneously exclaim: "I can't tell exactly who is credible!"
They each then think for a moment, and they realize that they can. If Tim and Ethan always tell the truth, then write on your answer sheet exactly which of the other three are credible.

p9. Pick an integer n between 1 and 10. If no other team picks the same number, we'll give you $\frac{n}{10}$ points.

p10. Many quantities in high-school mathematics are left undefined. Propose a definition or value for the following expressions and justify your response for each. We'll give you $\frac15$ points for each reasonable argument.

$$(i)\ \ (.5)! \quad (ii)\ \ \infty\cdot 0 \quad (iii)\ \ 0^0 \quad (iv)\ \ \prod_{x\in\emptyset} x \quad (v)\ \ 1-1+1-1+...$$

p11. On the back of your answer sheet, write the "coolest" math question you know, and include the solution. If the graders like your question the most, then you'll get a point. (With your permission, we might include your question on the Mixer next year!)

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