

Math Majors of America Tournament for High Schools 2015

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by parmenides51

– Mathathon Round

– Round 1

p1. If this mathathon has 7 rounds of 3 problems each, how many problems does it have in total? (Not a trick!)

p2. Five people, named $A, B, C, D,$ and $E,$ are standing in line. If they randomly rearrange themselves, what's the probability that nobody is more than one spot away from where they started?

p3. At Barrios's absurdly priced fish and chip shop, one fish is worth \$13, one chip is worth \$5. What is the largest integer dollar amount of money a customer can enter with, and not be able to spend it all on fish and chips?

Round 2

p4. If there are 15 points in 4-dimensional space, what is the maximum number of hyperplanes that these points determine?

p5. Consider all possible values of $\frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_1 - z_4}{z_2 - z_4}$ for any distinct complex numbers $z_1, z_2, z_3,$ and $z_4.$ How many complex numbers cannot be achieved?

p6. For each positive integer $n,$ let $S(n)$ denote the number of positive integers $k \leq n$ such that $\gcd(k, n) = \gcd(k + 1, n) = 1.$ Find $S(2015).$

Round 3

p7. Let $P_1, P_2, \dots, P_{2015}$ be 2015 distinct points in the plane. For any $i, j \in \{1, 2, \dots, 2015\},$ connect P_i and P_j with a line segment if and only if $\gcd(i - j, 2015) = 1.$ Define a clique to be a set of points such that any two points in the clique are connected with a line segment. Let ω be the unique positive integer such that there exists a clique with ω elements and such that there does not exist a clique with $\omega + 1$ elements. Find $\omega.$

p8. A Chinese restaurant has many boxes of food. The manager notices that • He can divide the boxes into groups of M where M is 19, 20, or 21. • There are exactly 3 integers x less than 16 such that grouping the boxes into groups of x leaves 3 boxes left over. Find the smallest possible number of boxes of food.

p9. If $f(x) = x|x| + 2$, then compute $\sum_{k=-1000}^{1000} f^{-1}(f(k) + f(-k) + f^{-1}(k))$.

Round 4

p10. Let ABC be a triangle with $AB = 13$, $BC = 20$, $CA = 21$. Let $ABDE$, $BCFG$, and $CAHI$ be squares built on sides AB , BC , and CA , respectively such that these squares are outside of ABC . Find the area of $DEHIFG$.

p11. What is the sum of all of the distinct prime factors of $7783 = 6^5 + 6 + 1$?

p12. Consider polyhedron $ABCDE$, where $ABCD$ is a regular tetrahedron and $BCDE$ is a regular tetrahedron. An ant starts at point A . Every time the ant moves, it walks from its current point to an adjacent point. The ant has an equal probability of moving to each adjacent point. After 6 moves, what is the probability the ant is back at point A ?

PS. You should use hide for answers. Rounds 5-7 have been posted here (<https://artofproblemsolving.com/community/c4h2782011p24434676>). Collected here (<https://artofproblemsolving.com/community/c5h2760506p24143309>).

- Round 5

p13. You have a 26×26 grid of squares. Color each randomly with red, yellow, or blue. What is the expected number (to the nearest integer) of 2×2 squares that are entirely red?

p14. Four snakes are boarding a plane with four seats. Each snake has been assigned to a different seat. The first snake sits in the wrong seat. Any subsequent snake will sit in their assigned seat if vacant, if not, they will choose a random seat that is available. What is the expected number of snakes who sit in their correct seats?

p15. Let $n \geq 1$ be an integer and $a > 0$ a real number. In terms of n , find the number of solutions (x_1, \dots, x_n) of the equation $\sum_{i=1}^n (x_i^2 + (a - x_i)^2) = na^2$ such that x_i belongs to the interval $[0, a]$, for $i = 1, 2, \dots, n$.

Round 6

p16. All roots of

$$\prod_{n=1}^{25} \prod_{k=0}^{2n} (-1)^k \cdot x^k = 0$$

are written in the form $r(\cos \phi + i \sin \phi)$ for $i^2 = -1$, $r > 0$, and $0 \leq \phi < 2\pi$. What is the smallest positive value of ϕ in radians?

p17. Find the sum of the distinct real roots of the equation

$$\sqrt[3]{x^2 - 2x + 1} + \sqrt[3]{x^2 - x - 6} = \sqrt[3]{2x^2 - 3x - 5}.$$

p18. If a and b satisfy the property that $a2^n + b$ is a square for all positive integers n , find all possible value(s) of a .

Round 7

p19. Compute $(1 - \cot 19^\circ)(1 - \cot 26^\circ)$.

p20. Consider triangle ABC with $AB = 3$, $BC = 5$, and $\angle ABC = 120^\circ$. Let point E be any point inside ABC . The minimum of the sum of the squares of the distances from E to the three sides of ABC can be written in the form a/b , where a and b are natural numbers such that the greatest common divisor of a and b is 1. Find $a + b$.

p21. Let $m \neq 1$ be a square-free number (an integer – possibly negative – such that no square divides m). We denote $Q(\sqrt{m})$ to be the set of all $a + b\sqrt{m}$ where a and b are rational numbers. Now for a fixed m , let S be the set of all numbers x in $Q(\sqrt{m})$ such that x is a solution to a polynomial of the form: $x^n + a_1x^{n-1} + \dots + a_n = 0$, where a_0, \dots, a_n are integers. For many integers m , $S = Z[\frac{m}{}] = \{a + b\sqrt{m}\}$ where a and b are integers. Give a classification of the integers for which this is not true. (Hint: It is true for $m = -1$ and 2 .)

PS. You should use hide for answers. Rounds 1-4 have been posted here (<https://artofproblemsolving.com/community/c4h2782002p24434611>). Collected here (<https://artofproblemsolving.com/community/c5h2760506p24143309>).

– Individual Round

- **p1.** The sum of two numbers, $a + b$, is 4 more than their difference, $a - b$. What is b ?
- p2.** Rectangle $ABCD$ has $AB = CD = \sqrt{3}$ and $AD = BC = 1$. Line ℓ is the perpendicular bisector of \overline{AC} and intersects \overline{AC} and \overline{AB} at P and Q , respectively. What is the area of quadrilateral $PQBC$?
- p3.** The polynomial $p(x) = x^3 + cx - 2$ has roots that are all integers. What is c ?
- p4.** There are 10 balls in a bucket, and there are 5 colors. Each color has exactly 2 balls of that color. Every time a ball is selected uniformly, randomly, and independently from the bucket, its color is noted and the ball is replaced. What is the expected number of selections from the bucket until one ball of every color has been seen?
- p5.** Consider a solid rectangular prism with length 10, width 8, and height 6. Find the volume of the set of points that are both a distance of at most 3 from the prism and a distance of at least 1 from the prism.
- p6.** Two positive integers, a and b are chosen randomly, uniformly, and independently from the set of positive integers less than 1000, $\{1, 2, \dots, 1000\}$. What is the expected value of the number of quadrants through which the graph $x^a + y^b = 1$ will pass?
- p7.** John and Mitchell are playing a game to see who gets the last of their candy. John rolls three unbiased 7-sided dice with sides labeled 1 through 7 and records the maximum roll. Mitchell flips a biased coin (with probability of heads p) 10 times and records the number of heads. What is the smallest p such that Mitchell has 50% or greater chance of winning the candy by recording a larger number?
- p8.** Define the sum of a finite set of integers to be the sum of the elements of the set. Let D be the set of positive divisors of 700. How many nonempty subsets of D have an even sum? (Simplify as reasonably as possible)
- p9.** Compute the absolute minimum of the function
- $$f(x) = \cos(2x) + 3 \cos(x)$$
- p10.** The largest prime factor of the number 520, 302, 325 has 5 digits. What is this prime factor?
- p11.** We play the following game with an equilateral triangle of $\frac{n(n+1)}{2}$ coins (with $n > 1$ coins

on each side). Initially, all of the coins are turned heads up. On each turn, we may turn over three coins that are mutually adjacent; the goal is to make all of the coins eventually turned tails up. What is the 7th smallest positive n for which this can be achieved?

p12. Let $S = \{a_1, a_2, a_3, \dots, a_{2015}\}$ be the first 2015 positive integers that a can be so that $2 + 2\sqrt{28a^2 + 1}$ is an integer. Compute the number of elements in S that a can be so that $2 + 2\sqrt{28a^2 + 1}$ is not a perfect square.

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– Tiebreaker Round

1 Each lattice point of the plane is labeled by a positive integer. Each of these numbers is the arithmetic mean of its four neighbors (above, below, left, right). Show that all the numbers are equal.

2 Determine, with proof, whether $22!6! + 1$ is prime.

3 Is there a number s in the set $\{\pi, 2\pi, 3\pi, \dots\}$ such that the first three digits after the decimal point of s are .001? Fully justify your answer.

4 For any nonnegative integer r , let S_r be a function whose domain is the natural numbers that satisfies

$$S_r(p^\alpha) = \begin{cases} 0 & \text{if } p \leq r \\ p^{\alpha-1}(p-r) & \text{if } p > r \end{cases}$$

for all primes p and positive integers α , and that $S_r(ab) = S_r(a)S_r(b)$ whenever a and b are relatively prime.

Now, suppose there are n squirrels at a party. Each squirrel is labeled with a unique number from the set $\{1, 2, \dots, n\}$. Two squirrels are friends with each other if and only if the difference between their labels is relatively prime to n . For example, if $n = 10$, then the squirrels with labels 3 and 10 are friends with each other because $10 - 3 = 7$, and 7 is relatively prime to 10.

Fix a positive integer m . Define a clique of size m to be any set of m squirrels at the party with the property that any two squirrels in the clique are friends with each other. Determine, with proof, a formula (using S_r) for the number of cliques of size m at the squirrel party.

– Mixer Round

Mixer Round p1. Let a_0, a_1, \dots, a_n be such that $a_n \neq 0$ and

$$(1 + x + x^3)^{341}(1 + 2x + x^2 + 2x^3 + 2x^4 + x^6)^{342} = \sum_{i=0}^n a_i x^i,$$

Find the number of odd numbers in the sequence $a_0; a_1; \dots; a_n$.

p2. Let $F_0 = 1$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

p3. Let a_n be the number of permutations of the numbers $S = \{1, 2, \dots, n\}$ such that for all k with $1 \leq k \leq n$, the sum of k and the number in the k th position of the permutation is a power of 2. Compute $a_{2^0} + a_{2^1} + \dots + a_{2^{20}}$.

p4. Three identical balls are painted white and black, so that half of each sphere is a white hemisphere, and the other half is a black one. The three balls are placed on a plane surface, each with a random orientation, so that each ball has a point of contact with the other two. What is the probability that at least one point of contact between two of the balls, both balls are the same color?

p5. Compute the greatest positive integer n such that there exists an odd integer a , for which $\frac{a^{2^n} - 1}{4^{4^n}}$ is not an integer.

p6. You are blind and cannot feel the difference between a coin that is heads up or tails up. There are 100 coins in front of you and are told that exactly 10 of them are heads up. On the back of this paper, explain how you can split the otherwise indistinguishable coins into two groups so that both groups have the same number of heads.

p7. On the back of this page, write the best math pun you can think of. You'll get a point if we chuckle.

p8. Pick an integer between 1 and 10. If you pick k , and n total teams pick k , then you'll receive $\frac{k}{10^n}$ points.

p9. There are four prisoners in a dungeon. Tomorrow, they will be separated into a group of three in one room, and the other in a room by himself. Each will be given a hat to wear that is either black or white – two will be given white and two black. None of them will be able to communicate with each other and none will see his or her own hat color. The group of three is lined up, so that the one in the back can see the other two, the second can see the first, but the first cannot see the others. If anyone is certain of their hat color, then they immediately shout that they know it to the rest of the group. If they can secretly prove it to the guard, they are saved. They only say something if they're sure. Which person is sure to survive?

p10. Down the road, there are 10 prisoners in a dungeon. Tomorrow they will be lined up in a single room and each given a black or white hat – this time they don't know how many of each. The person in the back can see everyone's hat besides his own, and similarly everyone else can only see the hats of the people in front of them. The person in the back will shout out a guess for his hat color and will be saved if and only if he is right. Then the person in front of him will have to guess, and this will continue until everyone has the opportunity to be saved. Each person can only say his or her guess of "white" or "black" when their turn comes, and no other signals may be made. If they have the night before receiving the hats to try to devise some sort of code, how many people at a minimum can be saved with the most optimal code? Describe the code on the back of this paper for full points.

p11. A few of the problems on this mixer contest were taken from last year's event. One of them had fewer than 5 correct answers, and most of the answers given were the same incorrect answer. Half a point will be given if you can guess the number of the problem on this test that corresponds to last year's question, and another .5 points will be given if you can guess the very common incorrect answer.

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