Math Majors of America Tournament for High Schools 2016

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by parmenides51

– Mathathon Round

– Round 1

p1. This year, the Mathathon consists of 7 rounds, each with 3 problems. Another math test, Aspartaime, consists of 3 rounds, each with 5 problems. How many more problems are on the Mathathon than on Aspartaime?

p2. Let the solutions to $x^3 + 7x^2 - 242x - 2016 = 0$be a, b, and c. Find $a^2 + b^2 + c^2$. (You might find it helpful to know that the roots are all rational.)

p3. For triangle ABC, you are given $AB = 8$ and $\angle A = 30^o$. You are told that BC will be chosen from amongst the integers from 1 to 10, inclusive, each with equal probability. What is the probability that once the side length BC is chosen there is exactly one possible triangle ABC?

Round 2

p4. It's raining! You want to keep your cat warm and dry, so you want to put socks, rain boots, and plastic bags on your cat's four paws. Note that for each paw, you must put the sock on before the boot, and the boot before the plastic bag. Also, the items on one paw do not affect the items you can put on another paw. How many different orders are there for you to put all twelve items of rain footwear on your cat?

p5. Let a be the square root of the least positive multiple of 2016 that is a square. Let b be the cube root of the least positive multiple of 2016 that is a cube. What is $a - b$?

p6. Hypersomnia Cookies sells cookies in boxes of $6, 9$ or 10. You can only buy cookies in whole boxes. What is the largest number of cookies you cannot exactly buy? (For example, you couldn't buy 8 cookies.)

Round 3

p7. There is a store that sells each of the 26 letters. All letters of the same type cost the same amount (i.e. any 'a' costs the same as any other 'a'), but different letters may or may not cost different amounts. For example, the cost of spelling "trade" is the same as the cost of spelling "tread," even though the cost of using a 't' may be different from the cost of an 'r.' If the letters to spell out 1 cost $\$1001$, the letters to spell out 2 cost $\$1010$, and the letters to spell out 11 cost $\$2015$, how much do the letters to spell out 12 cost?

p8. There is a square $ABCD$ with a point P inside. Given that $PA = 6$, $PB = 9$, $PC = 8$. Calculate PD.

p9. How many ordered pairs of positive integers (x, y) are solutions to $x^2 - y^2 = 2016$?

Round 4

p10. Given a triangle with side lengths $5, 6$ and 7, calculate the sum of the three heights of the triangle.

p11. There are 6 people in a room. Each person simultaneously points at a random person in the room that is not him/herself. What is the probability that each person is pointing at someone who is pointing back to them?

p12. Find all x such that $\sum_{i=0}^{\infty} ix^i = \frac{3}{4}$.

PS. You should use hide for answers. Rounds 5-7 have been posted here (`https://artofproblemsolving.com/community/c4h2782837p24446063`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Let $\{a\}_{n\ge 1}$ be an arithmetic sequence, with $a_1 = 0$, such that for some positive integers k and x we have $a_{k+1} = \binom{k}{x}$, $a_{2k+1} = \binom{k}{x+1}$, and $a_{3k+1} = \binom{k}{x+2}$. Let $\{b\}_{n\ge 1}$ be an arithmetic sequence of integers with $b_1 = 0$. Given that there is some integer m such that $b_m = \binom{k}{x}$, what is the number of possible values of b_2?

p14. Let $A = arcsin\left(\frac{1}{4}\right)$ and $B = arcsin\left(\frac{1}{7}\right)$. Find $\sin(A + B)\sin(A - B)$.

p15. Let $\{f_i\}_{i=1}^{9}$ be a sequence of continuous functions such that $f_i : R \to Z$ is continuous (i.e. each f_i maps from the real numbers to the integers). Also, for all i, $f_i(i) = 3^i$. Compute $\sum_{k=1}^{9} f_k \circ f_{k-1} \circ ... \circ f_1(3^{-k})$.

Round 6

p16. If x and y are integers for which $\frac{10x^3+10x^2y+xy^3+y^4}{203} = 1134341$ and $x - y = 1$, then compute $x + y$.

p17. Let T_n be the number of ways that n letters from the set $\{a, b, c, d\}$ can be arranged in a line (some letters may be repeated, and not every letter must be used) so that the letter a occurs an odd number of times. Compute the sum $T_5 + T_6$.

p18. McDonald plays a game with a standard deck of 52 cards and a collection of chips numbered 1 to 52. He picks 1 card from a fully shuffled deck and 1 chip from a bucket, and his score is the product of the numbers on card and on the chip. In order to win, McDonald must obtain a score that is a positive multiple of 6. If he wins, the game ends; if he loses, he eats a burger, replaces the card and chip, shuffles the deck, mixes the chips, and replays his turn. The probability that he wins on his third turn can be written in the form $\frac{x^2 \cdot y}{z^3}$ such that x, y, and z are relatively prime positive integers. What is $x + y + z$?

(NOTE: Use Ace as 1, Jack as 11, Queen as 12, and King as 13)

Round 7

p19. Let $f_n(x) = ln(1 + x^{2^n} + x^{2^{n+1}} + x^{3 \cdot 2^n})$. Compute $\sum_{k=0}^{\infty} f_{2k}\left(\frac{1}{2}\right)$.

p20. $ABCD$ is a quadrilateral with $AB = 183$, $BC = 300$, $CD = 55$, $DA = 244$, and $BD = 305$. Find AC.

p21. Define $\overline{xyz(t+1)} = 1000x + 100y + 10z + t + 1$ as the decimal representation of a four digit integer. You are given that $3^x 5^y 7^z 2^t = \overline{xyz(t+1)}$ where x, y, z, and t are non-negative integers such that t is odd and $0 \le x, y, z, (t+1) \le 9$. Compute$3^x 5^y 7^z$

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c4h2782822p24445934). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Round

– **p1.** For what value of x is the function $f(x) = (x-2)^2$ minimized?

p2. Two spheres A and B have centers $(0, 0, 0)$ and $(2016, 2016, 1008)$. Sphere A has a radius of 2017. If A and B are externally tangent, what is the radius of sphere B?

p3. Consider a white, solid cube of side length 5 made of $5 \times 5 \times 5 = 125$ identical unit cubes with faces parallel to the faces of the larger cube. The cube is submerged in blue paint until the entire exterior of the cube is painted blue, so that a face of a smaller cube is blue if and only if it is part of a face of the larger cube. A random smaller cube is selected and rolled. What is the probability that the up-facing side is blue?

p4. [This problem was thrown out.]

p5. Find the largest prime factor of 4003997, given that 4003997 is the product of two primes.

p6. Elaine is writing letters to six friends. She has six addressed letters and six addressed envelopes. She puts each letter randomly into an envelope without first checking the name on the envelope. What is the probability that exactly one envelope has the correct letter?

p7. Colin has written the numbers $1, 2, ..., n$ on a chalk board. He will erase at most 4 of the numbers (he might choose not to erase any of the numbers) and then circle $n - 4$ of the remaining numbers. There are exactly 2016 possible ways to do this. Find n. (You should assume that circling the same set of numbers but erasing different numbers should count as different possible ways.)

p8. Let $R_n = r_1, r_2, r_3, ..., r_n$ be a finite sequence of integers such that for all possible i, r_i is either -1, 0, 1 or 2. Furthermore, for all i such that $1 \le i < n$, r_i and r_{i+1} have opposite parity (i.e. one is odd and the other is even). Finally, -1 and 2 do not occur adjacently in the sequence. Given that r_1 must be even (i.e. either $r_1 = 0$ or $r_1 = 2$), $S(n)$ is the number of possible sequences R_n could be. For example, $S(1) = 2$. For what k is $S(k) = 12^2$?

p9. What is the largest prime p for which the numbers $p^2 - 8$, $p^2 - 2$, and $p^2 + 10$ are all prime as well?

p10. Express $\sqrt{25 + 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25}$ as an integer.

p11. Let $ABCD$ be a quadrilateral where AC bisects $\angle A$, $AB \ne AD$, $BC = CD = 7$, and

$AC \cdot BD = 36$. Find $AB + AD$.

p12. Find the largest integer x for which there is an integer y such that $x^4 + 12x^3 + 39x^2 + 17x - 57 = y^3$.

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– Tiebreaker Round

1 Let unit blocks be unit squares in the coordinate plane with vertices at lattice points (points (a, b) such that a and b are both integers). Prove that a circle with area π can cover parts of no more than 9 unit blocks. The circle below covers part of 8 unit blocks.
https://cdn.artofproblemsolving.com/attachments/4/4/43da9abed06d0feba94012ba68c177e3c2835
png

2 Suppose we have 2016 points in a 2-dimensional plane such that no three lie on a line. Two quadrilaterals are not disjoint if they share an edge or vertex, or if their edges intersect. Show that there are at least 504 quadrilaterals with vertices among these points such that any two of the quadrilaterals are disjoint.

3 Show that there are no integers x, y, z, and t such that

$$\sqrt[3]{x^5 + y^5 + z^5 + t^5} = 2016.$$

4 For real numbers a, b, c with $a + b + c = 3$, prove that

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \geq \frac{9}{2}abc(1 - abc)$$

and state when equality is reached.

– Mixer Round

Mixer Round p1. Give a fake proof that $0 = 1$ on the back of this page. The most convincing answer to this question at this test site will receive a point.

p2. It is often said that once you assume something false, anything can be derived from it. You may assume for this question that $0 = 1$, but you can only use other statements if they are generally accepted as true or if your prove them from this assumption and other generally acceptable mathematical statements. With this in mind, on the back of this page prove that every number is the same number.

p3. Suppose you write out all integers between 1 and 1000 inclusive. (The list would look something like 1, 2, 3, ... , 10, 11, ... , 999, 1000.) Which digit occurs least frequently?

p4. Pick a real number between 0 and 1 inclusive. If your response is r and the standard deviation of all responses at this site to this question is σ, you will receive $r(1-(r-\sigma)^2)$ points.

p5. Find the sum of all possible values of x that satisfy $243^{x+1} = 81^{x^2+2x}$.

p6. How many times during the day are the hour and minute hands of a clock aligned?

p7. A group of $N + 1$ students are at a math competition. All of them are wearing a single hat on their head. N of the hats are red; one is blue. Anyone wearing a red hat can steal the blue hat, but in the process that person's red hat disappears. In fact, someone can only steal the blue hat if they are wearing a red hat. After stealing it, they would wear the blue hat. Everyone prefers the blue hat over a red hat, but they would rather have a red hat than no hat at all. Assuming that everyone is perfectly rational, find the largest prime N such that nobody will ever steal the blue hat.

p8. On the back of this page, prove there is no function f(x) for which there exists a (finite degree) polynomial $p(x)$ such that $f(x) = p(x)(x + 3) + 8$ and $f(3x) = 2f(x)$.

p9. Given a cyclic quadrilateral $YALE$ with $YA = 2$, $AL = 10$, $LE = 11$, $EY = 5$, what is the area of $YALE$?

p10. About how many pencils are made in the U.S. every year? If your answer to this question is p, and our (good) estimate is ρ, then you will receive $\max(0, 1-\frac{1}{2}|\log_{10}(p)-\log_{10}(\rho)|)$ points.

p11. The largest prime factor of $520, 302, 325$ has 5 digits. What is this prime factor?

p12. The previous question was on the individual round from last year. It was one of the least frequently correctly answered questions. The first step to solving the problem and spotting the pattern is to divide $520, 302, 325$ by an appropriate integer. Unfortunately, when solving the problem many people divide it by n instead, and then they fail to see the pattern. What is n?

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