Japan MO Finals 2022

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1 There are 2022 grids in a row. Two people A and B play a game with these grids. At first, they mark each odd-numbered grid from the left with A's name, and each even-numbered grid from the left with B's name. Then, starting with the player A, they take turns performing the following action:

• One should select two grids marked with his/her own name, such that these two grids are not adjacent, and all grids between them are marked with the opponent's name. Then, change the name in all the grids between those two grids to one's own name.

Two people take turns until one can not perform anymore. Find the largest positive integer m satisfying the following condition:

• No matter how B acts, A can always make sure that there are at least m grids marked with A's name.

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2 Find all functions $f:\mathbb{Z}_+\to\mathbb{Z}_+$, such that for any two positive integers m,n, we have

$$f^{f(n)}(m)+mn=f(m)f(n)$$

where $f^k(n)=\underbrace{f(f(\ldots f(}_{k}n)\ldots))$.

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3 In an isosceles triangle ABC, $AB=AC$. Take a point O inside the triangle (not on the boundary). Draw a circle ω with center O and passing through C. Suppose $\omega\cap BC=\{C,D\}$, $\omega\cap AC=\{C,E\}$. Denote the circumcircle of $\triangle AEO$ as Γ. And suppose $\Gamma\cap\omega=\{E,F\}$. Prove that the circumcenter of $\triangle BDF$ is on Γ.

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4 Find all positive integer pairs (x,y) satisfying the following equation:

$$3^x-8^y=2xy+1$$

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5 Find the smallest positive integer m satisfying the following proposition:

There are 999 grids on a circle. Fill a real number in each grid, such that for any grid A and any positive integer $k\le m$, at least one of the following two propositions will be true:

• The difference between the numbers in the grid A and the k-th grid after A in clockwise direction is k; • The difference between the numbers in the grid A and the k-th grid after A in anticlockwise direction is k.

Then, there must exist a grid S with the real number x in it on the circle, such that at least one of the following two propositions will be true:

• For any positive integer $k < 999$, the number in the k-th grid after S in clockwise direction is $x + k$; • For any positive integer $k < 999$, the number in the k-th grid after S in anticlockwise direction is $x + k$.