International Zhautykov Olympiad 2022

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by mathematics2004, oVlad

1 Non-zero polynomials $P(x)$, $Q(x)$, and $R(x)$ with real coefficients satisfy the identities

$$P(x) + Q(x) + R(x) = P(Q(x)) + Q(R(x)) + R(P(x)) = 0.$$

Prove that the degrees of the three polynomials are all even.

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2 A ten-level 2-tree is drawn in the plane: a vertex A_1 is marked, it is connected by segments with two vertices B_1 and B_2, each of B_1 and B_2 is connected by segments with two of the four vertices C_1, C_2, C_3, C_4 (each C_i is connected with one B_j exactly); and so on, up to 512 vertices $J_1, \ldots, J_{512}$. Each of the vertices $J_1, \ldots, J_{512}$ is coloured blue or golden. Consider all permutations f of the vertices of this tree, such that (i) if X and Y are connected with a segment, then so are $f(X)$ and $f(Y)$, and (ii) if X is coloured, then $f(X)$ has the same colour. Find the maximum M such that there are at least M permutations with these properties, regardless of the colouring.

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3 In parallelogram $ABCD$ with acute angle A a point N is chosen on the segment AD, and a point M on the segment CN so that $AB = BM = CM$. Point K is the reflection of N in line MD. The line MK meets the segment AD at point L. Let P be the common point of the circumcircles of AMD and CNK such that A and P share the same side of the line MK. Prove that $\angle CPM = \angle DPL$.

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4 In triangle ABC, a point M is the midpoint of AB, and a point I is the incentre. Point A_1 is the reflection of A in BI, and B_1 is the reflection of B in AI. Let N be the midpoint of A_1B_1. Prove that $IN > IM$.

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5 A polynomial $f(x)$ with real coefficients of degree greater than 1 is given. Prove that there are infinitely many positive integers which cannot be represented in the form

$$f(n+1) + f(n+2) + \cdots + f(n+k)$$

where n and k are positive integers.

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6 Do there exist two bounded sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for each positive integers n and $m > n$ at least one of the two inequalities $|a_m - a_n| > 1/\sqrt{n}$, and $|b_m - b_n| > 1/\sqrt{n}$ holds?
