Math Majors of America Tournament for High Schools 2017

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by parmenides51

– Mathathon Round

– Round 1

p1. Jom and Terry both flip a fair coin. What is the probability both coins show the same side?

p2. Under the same standard air pressure, when measured in Fahrenheit, water boils at 212^o F and freezes at 32^o F. At thesame standard air pressure, when measured in Delisle, water boils at 0 D and freezes at 150 D. If x is today's temperature in Fahrenheit and y is today's temperature expressed in Delisle, we have $y = ax + b$. What is the value of $a + b$? (Ignore units.)

p3. What are the last two digits of $5^1 + 5^2 + 5^3 + + 5^{10} + 5^{11}$?

Round 2

p4. Compute the average of the magnitudes of the solutions to the equation $2x^4 + 6x^3 + 18x^2 + 54x + 162 = 0$.

p5. How many integers between 1 and 1000000 inclusive are both squares and cubes?

p6. Simon has a deck of 48 cards. There are 12 cards of each of the following 4 suits: fire, water, earth, and air. Simon randomly selects one card from the deck, looks at the card, returns the selected card to the deck, and shuffles the deck. He repeats the process until he selects an air card. What is the probability that the process ends without Simon selecting a fire or a water card?

Round 3

p7. Ally, Beth, and Christine are playing soccer, and Ally has the ball. Each player has a decision: to pass the ball to a teammate or to shoot it. When a player has the ball, they have a probability p of shooting, and $1-p$ of passing the ball. If they pass the ball, it will go to one of the other two teammates with equal probability. Throughout the game, p is constant. Once the ball has been shot, the game is over. What is the maximum value of p that makes Christine's total probability of shooting the ball $\frac{3}{20}$?

p8. If x and y are real numbers, then what is the minimum possible value of the expression $3x^2 - 12xy + 14y^2$ given that $x - y = 3$?

p9. Let ABC be an equilateral triangle, let D be the reflection of the incenter of triangle ABC over segment AB, and let E be the reflection of the incenter of triangle ABD over segment AD. Suppose the circumcircle Ω of triangle ADE intersects segment AB again at X. If the length of AB is 1, find the length of AX.

Round 4

p10. Elaine has c cats. If she divides c by 5, she has a remainder of 3. If she divides c by 7, she has a remainder of 5. If she divides c by 9, she has a remainder of 7. What is the minimum value c can be?

p11. Your friend Donny offers to play one of the following games with you. In the first game, he flips a fair coin and if it is heads, then you win. In the second game, he rolls a 10-sided die (its faces are numbered from 1 to 10) x times. If, within those x rolls, the number 10 appears, then you win. Assuming that you like winning, what is the highest value of x where you would prefer to play the coin-flipping game over the die-rolling game?

p12. Let be the set $X = \{0, 1, 2, ..., 100\}$. A subset of X, called N, is defined as the set that contains every element x of X such that for any positive integer n, there exists a positive integer k such that n can be expressed in the form $n = x^{a_1} + x^{a_2} + ... + x^{a_k}$, for some integers $a_1, a_2, ..., a_k$ that satisfy $0 \le a_1 \le a_2 \le ... \le a_k$. What is the sum of the elements in N?

PS. You should use hide for answers. Rounds 5-7 have be posted here (`https://artofproblemsolving.com/community/c4h2782880p24446580`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Points A, B, C, and D lie in a plane with $AB = 6$, $BC = 5$, and $CD = 5$, and AB is perpendicular to BC. Point E lies on line AD such that $D \ne E$, $AE = 3$ and $CE = 5$. Find DE.

p14. How many ordered pairs of integers (x, y) are solutions to $x^2y = 36 + y$?

p15. Chicken nuggets come in boxes of two sizes, a nuggets per box and b nuggets per box. We know that 899 nuggets is the largest number of nuggets we cannot obtain with some combination of a-sized boxes and b-sized boxes. How many different pairs (a, b) are there with $a < b$?

Round 6

p16. You are playing a game with coins with your friends Alice and Bob. When all three of you flip your respective coins, the majority side wins. For example, if Alice, Bob, and you flip Heads, Tails, Heads in that order, then you win. If Alice, Bob, and you flip Heads, Heads, Tails in that order, then you lose. Notice that more than one person will "win." Alice and Bob design their coins as follows: a value p is chosen randomly and uniformly between 0 and 1. Alice then makes a biased coin that lands on heads with probability p, and Bob makes a biased coin that lands on heads with probability $1-p$. You design your own biased coin to maximize your chance of winning without knowing p. What is the probability that you win?

p17. There are N distinct students, numbered from 1 to N. Each student has exactly one hat: y students have yellow hats, b have blue hats, and r have red hats, where $y+b+r=N$ and $y, b, r > 0$. The students stand in a line such that all the r people with red hats stand in front of all the b people with blue hats. Anyone wearing red is standing in front of everyone wearing blue. The y people with yellow hats can stand anywhere in the line. The number of ways for the students to stand in a line is 2016. What is $100y + 10b + r$?

p18. Let P be a point in rectangle $ABCD$ such that $\angle APC = 135^o$ and $\angle BPD = 150^o$. Suppose furthermore that the distance from P to AC is 18. Find the distance from P to BD.

Round 7

p19. Let triangle ABC be an isosceles triangle with $|AB| = |AC|$. Let D and E lie on AB and AC, respectively. Suppose $|AD| = |BC| = |EC|$ and triangle ADE is isosceles. Find the sum of all possible values of $\angle BAC$ in radians. Write your answer in the form $2arcsin\left(\frac{a}{b}\right)+\frac{c}{d}\pi$, where $\frac{a}{b}$ and $\frac{c}{d}$ are in lowest terms, $-1 \le \frac{a}{b} \le 1$, and $-1 \le \frac{c}{d} \le 1$.

p20. Kevin is playing a game in which he aims to maximize his score. In the n^{th} round, for $n \ge 1$, a real number between 0 and $\frac{1}{3^n}$ is randomly generated. At each round, Kevin can either choose to have the randomly generated number from that round as his score and end the game, or he can choose to pass on the number and continue to the next round. Once Kevin passes on a number, he CANNOT claim that number as his score. Kevin may continue playing for as many rounds as he wishes. If Kevin plays optimally, the expected value of his score is $a+b\sqrt{c}$ where a, b, and c are integers and c is positive and not divisible by any positive perfect square other than 1. What is $100a + 10b + c$?

p21. Lisa the ladybug (a dimensionless ladybug) lives on the coordinate plane. She begins at the origin and walks along the grid, at each step moving either right or up one unit. The path she takes ends up at $(2016, 2017)$. Define the "area" of a path as the area below the path and above the x-axis. The sum of areas over all paths that Lisa can take can be represented as as $a \cdot \binom{4033}{2016}$. What is the remainder when a is divided by 1000?

PS. You should use hide for answers. Rounds 1-4 have been posted here (`https://artofproblemsolving.com/community/c4h2782871p24446475`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Round

– **p1.** What is the smallest positive prime divisor of $101^2 - 99^2$?

p2. The product $81 \cdot 13579$ equals $1099A99$ for some digit A. What is A?

p3. On a MMATHS team of 6 students, all students forget which of the 5 MMATHS sites (Yale, UVA, UF, UM, and Columbia) they are supposed to attend. On competition day, each student chooses a random site uniformly and independently of each other to attend, and they each take a flight to the one they picked. What is the probability that all of the sites end up with at least one person from that team?

p4. Square $ABCD$ has area 12. Let E be the midpoint of AD, and let F be the point of intersection of BE and AC. Find the area of quadrilateral $EFCD$.

p5. Alexander is eating sliced olives for dinner. Olives are sliced into thirds so that each olive is composed of two indistinguishable "end" pieces and one "middle" piece. For each olive slice that Alexander puts on his plate, there is a $\frac{2}{3}$ chance that it is an end piece and a $\frac{1}{3}$ chance that it is a middle piece. After placing a randomly selected olive slice on his plate, he checks to see if he can rearrange the slices on his plate so that he has at least one whole olive, which consists of any middle piece and any two end pieces. He can successfully rearrange the olive slices on his plate into at least one whole olive after randomly selecting n olive slices with probability of at least 0.8. What is the minimum n for which this is true?

p6. What is the smallest positive integer n whose digits multiply to 8000?

p7. Mitchell has a bunch of 2×1 rectangular tiles. He needs to arrange them in such a way that they exactly cover a 2×10 grid of 1×1 squares. How many ways are there for him to do this?

Two such "tilings" are considered distinct from each other if not all of the 2-by-1 rectangular tiles in the two tilings are placed in exactly the same position and orientation.

p8. Trapezoid $ABCD$ is inscribed in a circle of radius 100. If $AB = 200$ and $BC = 50$, what is the perimeter of $ABCD$?

p9. How many ordered pairs of nonzero integers (a, b) are there such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{24}$?

p10. Suppose a and b are complex numbers such that $|a| = |b| = 1$ and $a + b = \frac{1}{3} + \frac{1}{4}i$. Find the product ab.

p11. Find the number of ordered triples of positive integers (a_1, a_2, a_3) such that $a_1 + a_2 + a_3 = 2017$ and 2 does not divide a_1, 3 does not divide a_2, and 4 does not divide a_3.

p12. Circles Γ and Ω have radii of 20 and 17, respectively, and their centers are separated by a distance of 5. Let X be a fixed intersection point of these two circles, and let Z be some point on Γ for which segment XZ intersects Ω at a second point Y . (If XZ is tangent to Ω, then define Y to be X.) Find the maximum possible length of segment YZ.

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–	Tiebreaker Round
1	For any integer $n > 4$, prove that $2^n > n^2$.
2	Suppose you are playing a game against Daniel. There are 2017 chips on a table. During your turn, if you can write the number of chips on the table as a sum of two cubes of not necessarily distinct, nonnegative integers, then you win. Otherwise, you can take some number of chips between 1 and 6 inclusive off the table. (You may not leave fewer than 0 chips on the table.) Daniel can also do the same on his turn. You make the first move, and you and Daniel always make the optimal move during turns. Who should win the game? Explain.
3	Let $f : R \to R$, and let P be a nonzero polynomial with degree no more than 2015. For any nonnegative integer n, $f^{(n)}(x)$ denotes the function defined as f composed with itself n times. For example, $f^{(0)}(x) = x$, $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = f(f(x))$, etc. Show that there always exists a real number q such that $$f^{((2017^{2017})!)(q)} \neq (q + 2017)(qP(q) - 1).$$

4 In a triangle ABC, let A_0 be the point where the interior angle bisector of angle A meets with side BC. Similarly define B_0 and C_0. Prove that $\angle B_0A_0C_0 = 90^o$ if and only if $\angle BAC = 120^o$.

– Mixer Round

Mixer Round **p1.** Suppose Mitchell has a fair die. He is about to roll it six times. The probability that he rolls 1, 2, 3, 4, 5, and then 6 in that order is p. The probability that he rolls 2, 2, 4, 4, 6, and then 6 in that order is q. What is $p - q$?

p2. What is the smallest positive integer x such that $x \equiv 2017$ (mod 2016) and $x \equiv 2016$ (mod 2017) ?

p3. The vertices of triangle ABC lie on a circle with center O. Suppose the measure of angle ACB is 45^o. If $|AB| = 10$, then what is the distance between O and the line AB?

p4. A "word" is a sequence of letters such as $YALE$ and $AELY$. How many distinct 3-letter words can be made from the letters in $BOOLABOOLA$ where each letter is used no more times than the number of times it appears in $BOOLABOOLA$?

p5. How many distinct complex roots does the polynomial $p(x) = x^{12} - x^8 - x^4 + 1$ have?

p6. Notice that $1 = \frac12 + \frac13 + \frac16$, that is, 1 can be expressed as the sum of the three fractions $\frac12$, $\frac13$, and $\frac16$, where each fraction is in the form $\frac1n$, with each n different. Give a 6-tuple of distinct positive integers (a, b, c, d, e, f) where $a < b < c < d < e < f$ such that $\frac1a + \frac1b + \frac1c + \frac1d + \frac1e + \frac1f = 1$ and explain how you arrived at your 6-tuple. Multiple answers will be accepted.

p7. You have a Monopoly board, an 11×11 square grid with the 9×9 internal square grid removed, where every square is blank except for Go, which is the square in the bottom right corner. During your turn, you determine how many steps forward (which is in the counterclockwise direction) to move by rolling two standard 6-sided dice. Let S be the set of squares on the board such that if you are initially on a square in S, no matter what you roll with the dice, you will always either land on Go (move forward enough squares such that you end up on Go) or you pass Go (you move forward enough squares such that you step on Go during your move and then you advance past Go). You randomly and uniformly select one square in S as your starting position. What is the probability that you land on Go?

p8. Using L-shaped triominos, and dominos, where each square of a triomino and a domino covers one unit, what is the minimum number of tiles needed to cover a 3-by-2017 rectangle without any gaps?

p9. Does there exist a pair of positive integers (x, y), where $x < y$, such that $x^2 + y^2 = 1009^3$? If so, give a pair (x, y) and explain how you found that pair. If not, explain why.

p10. Triangle ABC has inradius 8 and circumradius 20. Let M be the midpoint of side BC, and let N be the midpoint of arc BC on the circumcircle not containing A. Let s_A denote the length of segment MN, and define s_B and s_C similarly with respect to sides CA and AB. Evaluate the product $s_As_Bs_C$.

p11. Julia and Dan want to divide up 256 dollars in the following way: in the first round, Julia will offer Dan some amount of money, and Dan can choose to accept or reject the offer. If Dan accepts, the game is over. Otherwise, if Dan rejects, half of the money disappears. In the second round, Dan can offer Julia part of the remaining money. Julia can then choose to accept or reject the offer. This process goes on until an offer is accepted or until 4 rejections have been made; once 4 rejections are made, all of the money will disappear, and the bargaining process ends. If Julia or Dan is indifferent between accepting and rejecting an offer, they will accept the offer. Given that Julia and Dan are both rational and both have the goal of maximizing the amount of money they get, how much will Julia offer Dan in the first round?

p12. A perfect partition of a positive integer N is an unordered set of numbers (where numbers can be repeated) that sum to N with the property that there is a unique way to express each positive integer less than N as a sum of elements of the set. Repetitions of elements of the set are considered identical for the purpose of uniqueness. For example, the only perfect partitions of 3 are $\{1, 1, 1\}$ and $\{1, 2\}$. $\{1, 1, 3, 4\}$ is NOT a perfect partition of 9 because the sum 4 can be achieved in two different ways: 4 and $1 + 3$. How many integers $1 \le N \le 40$ each have exactly one perfect partition?

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