BdMO 2021 Higher Secondary National Round problems

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Problem 1 For a positive integer n, let $A(n)$ be the equal to the remainder when n is divided by 11 and let $T(n) = A(1) + A(2) + A(3) + \cdots + A(n)$. Find the value of

$$A(T(2021))$$

Problem 2 Let u, v be real numbers. The minimum value of $\sqrt{u^2+v^2}+\sqrt{(u-1)^2+v^2}+\sqrt{u^2+(v-1)^2}+\sqrt{(u-1)^2+(v-1)^2}$ can be written as $\sqrt{n}$. Find the value of $10n$.

Problem 3 Let ABC be a triangle with incenter I. Points E and F are on segments AC and BC respectively such that, $AE = AI$ and $BF = BI$. If EF is the perpendicular bisector of CI, then $\angle ACB$ in degrees can be written as $\frac{m}{n}$ where m and n are co-prime positive integers. Find the value of $m+n$.

Problem 4 $P(x)$ is a polynomial in x with non-negative integer coefficients. If $P(1) = 5$ and $P(P(1)) = 177$, what is the sum of all possible values of $P(10)$?

Problem 5 How many ways can you roll three 20-sided dice such that the sum of the three rolls is exactly 42? Here the order of the rolls matter. [i](Note that a 20-sided die is is very much like a regular 6-sided die other than the fact that it has 20 faces instead of 6)[/i]

Problem 6 Let ABC be an acute-angled triangle. The external bisector of $\angle BAC$ meets the line BC at point N. Let M be the midpoint of BC. P and Q are two points on line AN such that, $\angle PMN = \angle MQN = 90°$. If $PN = 5$ and $BC = 3$, then the length QA can be expressed as $\frac{a}{b}$ where a and b are co-prime positive integers. What is the value of $(a+b)$?

Problem 7 A binary string is a word containing only 0s and 1s. In a binary string, a $1-$run is a non extendable substring containing only 1s. Given a positive integer n, let $B(n)$ be the number of $1-$runs in the binary representation of n. For example, $B(107) = 3$ since 107 in binary is 1101011 which has exactly three $1-$runs. What is the following expression equal to?

$$B(1) + B(2) + B(3) + \cdots + B(255)$$

Problem 8 Shakur and Tiham are playing a game. Initially, Shakur picks a positive integer not greater than 1000. Then Tiham picks a positive integer strictly smaller than that.Then they keep on doing this taking turns to pick progressively smaller and smaller positive integers until some one picks 1. After that, all the numbers that have been picked so far are added up. The person picking the number 1 wins if and only if this sum is a perfect square. Otherwise, the other player wins. What is the sum of all possible values of n such that if Shakur starts with the number n, he has a winning strategy?

Problem 9 A positive integer n is called nice if it has at least 3 proper divisors and it is equal to the sum of its three largest proper divisors. For example, 6 is nice because its largest proper divisors are $3, 2, 1$ and $6 = 3 + 2 + 1$. Find the number of nice integers not greater than 3000.

Problem 10 $A_1A_2A_3A_4A_5A_6A_7A_8$ is a regular octagon. Let P be a point inside the octagon such that the distances from P to A_1A_2, A_2A_3 and A_3A_4 are $24, 26$ and 27 respectively. The length of A_1A_2 can be written as $a\sqrt{b}-c$, where a, b and c are positive integers and b is not divisible by any square number other than 1. What is the value of $(a + b + c)$?

Problem 11 How many quadruples of positive integers (a, b, m, n) are there such that all of the following statements hold?

1. $a, b < 5000$
2. $m, n < 22$
3. $gcd(m, n) = 1$
4. $(a^2 + b^2)^m = (ab)^n$

Problem 12 A function $g : \mathbb{Z} \to \mathbb{Z}$ is called adjective if $g(m) + g(n) > max(m^2, n^2)$ for any pair of integers m and n. Let f be an adjective function such that the value of $f(1) + f(2) + \cdots + f(30)$ is minimized. Find the smallest possible value of $f(25)$.
