**Math Majors of America Tournament for High Schools 2018**

www.artofproblemsolving.com/community/c2995167

by parmenides51

– Mathathon Round

– Round 1

**p1.** Elaine creates a sequence of positive integers $\{s_n\}$. She starts with $s_1 = 2018$. For $n \ge 2$, she sets $s_n = \frac12 s_{n-1}$ if $s_{n-1}$ is even and $s_n = s_{n-1} + 1$ if $s_{n-1}$ is odd. Find the smallest positive integer $n$ such that $s_n = 1$, or submit "$0$" as your answer if no such $n$ exists.

**p2.** Alice rolls a fair six-sided die with the numbers $1$ through $6$, and Bob rolls a fair eight-sided die with the numbers $1$ through $8$. Alice wins if her number divides Bob's number, and Bob wins otherwise. What is the probability that Alice wins?

**p3.** Four circles each of radius $\frac14$ are centered at the points $\left(\pm \frac14, \pm \frac14\right)$, and ther exists a fifth circle is externally tangent to these four circles. What is the radius of this fifth circle?

Round 2

**p4.** If Anna rows at a constant speed, it takes her two hours to row her boat up the river (which flows at a constant rate) to Bob's house and thirty minutes to row back home. How many minutes would it take Anna to row to Bob's house if the river were to stop flowing?

**p5.** Let $a_1 = 2018$, and for $n \ge 2$ define $a_n = 2018^{a_{n-1}}$ . What is the ones digit of $a_{2018}$?

**p6.** We can write $(x + 35)^n = \sum_{i=0}^{n} c_i x^i$ for some positive integer $n$ and real numbers $c_i$. If $c_0 = c_2$, what is $n$?

Round 3

**p7.** How many positive integers are factors of $12!$ but not of $(7!)^2$?

**p8.** How many ordered pairs $(f(x), g(x))$ of polynomials of degree at least $1$ with integer coefficients satisfy $f(x)g(x) = 50x^6 - 3200$?

**p9.** On a math test, Alice, Bob, and Carol are each equally likely to receive any integer score between $1$ and $10$ (inclusive). What is the probability that the average of their three scores is an integer?

Round 4

**p10.** Find the largest positive integer N such that

$$(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)$$

is divisible by $N$ for all choices of positive integers $a > b > c > d > e$.

**p11.** Let $ABCDE$ be a square pyramid with $ABCD$ a square and E the apex of the pyramid. Each side length of $ABCDE$ is $6$. Let $ABCDD'C'B'A'$ be a cube, where $AA'$, $BB'$, $CC'$, $DD'$ are edges of the cube. Andy the ant is on the surface of $EABCDD'C'B'A'$ at the center of triangle $ABE$ (call this point $G$) and wants to crawl on the surface of the cube to $D'$. What is the length the shortest path from $G$ to $D'$? Write your answer in the form $\sqrt{a+b\sqrt{3}}$, where $a$ and $b$ are positive integers.

**p12.** A six-digit palindrome is a positive integer between $100,000$ and $999,999$ (inclusive) which is the same read forwards and backwards in base ten. How many composite six-digit palindromes are there?

PS. You should use hide for answers. Rounds 5-7 have been posted here (https://artofproblemsolving.com/community/c4h2784943p24473026). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 5

**p13.** Circles $\omega_1$, $\omega_2$, and $\omega_3$ have radii $8$, $5$, and $5$, respectively, and each is externally tangent to the other two. Circle $\omega_4$ is internally tangent to $\omega_1$, $\omega_2$, and $\omega_3$, and circle $\omega_5$ is externally tangent to the same three circles. Find the product of the radii of $\omega_4$ and $\omega_5$.

**p14.** Pythagoras has a regular pentagon with area $1$. He connects each pair of non-adjacent vertices with a line segment, which divides the pentagon into ten triangular regions and one pentagonal region. He colors in all of the obtuse triangles. He then repeats this process using the smaller pentagon. If he continues this process an infinite number of times, what is the total area that he colors in? Please rationalize the denominator of your answer.

p15. Maisy arranges $61$ ordinary yellow tennis balls and $3$ special purple tennis balls into a $4\times 4\times 4$ cube. (All tennis balls are the same size.) If she chooses the tennis balls' positions in the cube randomly, what is the probability that no two purple tennis balls are touching?

Round 6

**p16.** Points $A, B, C$, and $D$ lie on a line (in that order), and $\triangle\ BCE$ is isosceles with $\overline{BE} = \overline{CE}$. Furthermore, $F$ lies on $\overline{BE}$ and $G$ lies on $\overline{CE}$ such that $\triangle\ BFD$ and $\triangle\ CGA$ are both congruent to $\triangle\ BCE$. Let $H$ be the intersection of $\overline{DF}$ and $\overline{AG}$, and let $I$ be the intersection of $\overline{BE}$ and $\overline{AG}$. If $m\angle BCE = arcsin\left(\frac{12}{13}\right)$, what is $\frac{HI}{FI}$ ?

**p17.** Three states are said to form a tri-state area if each state borders the other two. What is the maximum possible number of tri-state areas in a country with fifty states? Note that states must be contiguous and that states touching only at "corners" do not count as bordering.

**p18.** Let $a, b, c, d$, and $e$ be integers satisfying

$$2(\sqrt[3]{2})^2 + \sqrt[3]{2}a + 2b + (\sqrt[3]{2})^2 c + \sqrt[3]{2}d + e = 0$$

and

$$25\sqrt{5}i + 25a - 5\sqrt{5}ib - 5c + \sqrt{5}id + e = 0$$

where $i = \sqrt{-1}$. Find $|a + b + c + d + e|$.

Round 7

**p19.** What is the greatest number of regions that $100$ ellipses can divide the plane into? Include the unbounded region.

**p20.** All of the faces of the convex polyhedron $P$ are congruent isosceles (but NOT equilateral) triangles that meet in such a way that each vertex of the polyhedron is the meeting point of either ten base angles of the faces or three vertex angles of the faces. (An isosceles triangle has two base angles and one vertex angle.) Find the sum of the numbers of faces, edges, and vertices of $P$.

**p21.** Find the number of ordered $2018$-tuples of integers $(x_1, x_2, ....x_{2018})$, where each integer is between $-2018^2$ and $2018^2$ (inclusive), satisfying

$$6(1x_1 + 2x_2 + ... + 2018x_{2018})^2 \ge (2018)(2019)(4037)(x_1^2 + x_2^2 + ... + x_{2018}^2).$$

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c4h2784936p24472982). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Round

– **p1.** Five friends arrive at a hotel which has three rooms. Rooms $A$ and $B$ hold two people each, and room $C$ holds one person. How many different ways could the five friends lodge for the night?

**p2.** The set of numbers $\{1, 3, 8, 12, x\}$ has the same average and median. What is the sum of all possible values of $x$? (Note that $x$ is not necessarily greater than $12$.)

**p3.** How many four-digit numbers $\overline{ABCD}$ are there such that the three-digit number $\overline{BCD}$ satisfies $\overline{BCD} = \frac{1}{6}\overline{ABCD}$? (Note that $A$ must be nonzero.)

**p4.** Find the smallest positive integer $n$ such that $n$ leaves a remainder of $5$ when divided by $14$, $n^2$ leaves a remainder of $1$ when divided by $12$, and $n^3$ leaves a remainder of $7$ when divided by $10$.

**p5.** In rectangle $ABCD$, let $E$ lie on $\overline{CD}$, and let $F$ be the intersection of $\overline{AC}$ and $\overline{BE}$. If the area of $\triangle\ ABF$ is $45$ and the area of $\triangle\ CEF$ is $20$, find the area of the quadrilateral $ADEF$.

**p6.** If $x$ and $y$ are integers and $14x^2y^3 - 38x^2 + 21y^3 = 2018$, what is the value of $x^2y$?

**p7.** $A, B, C, D$ all lie on a circle with $\overline{AB} = \overline{BC} = \overline{CD}$. If the distance between any two of these points is a positive integer, what is the smallest possible perimeter of quadrilateral $ABCD$?

**p8.** Compute

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\cos^2(n) + n\sin^2(m)}{3^{m+n}(m+n)}$$

**p9.** Diane has a collection of weighted coins with different probabilities of landing on heads, and she flips nine coins sequentially according to a particular set of rules. She uses a coin that always lands on heads for her first and second flips, and she uses a coin that always lands on tails for her third flip. For each subsequent flip, she chooses a coin to flip as follows: if she has so far flipped $a$ heads out of $b$ total flips, then she uses a coin with an $\frac{a}{b}$ probability of landing on heads. What is the probability that after all nine flips, she has gotten six heads and three tails?

**p10.** For any prime number $p$, let $S_p$ be the sum of all the positive divisors of $37^p p^{37}$ (including $1$ and $37^p p^{37}$). Find the sum of all primes $p$ such that $S_p$ is divisible by $p$.

**p11.** Six people are playing poker. At the beginning of the game, they have $1, 2, 3, 4, 5$, and $6$ dollars, respectively. At the end of the game, nobody has lost more than a dollar, and each player has a distinct nonnegative integer dollar amount. (The total amount of money in the game remains constant.) How many distinct finishing rankings (i.e. lists of first place through sixth place) are possible?

**p12.** Let $C_1$ be a circle of radius $1$, and let $C_2$ be a circle of radius $\frac{1}{2}$ internally tangent to $C_1$. Let $\{\omega_0, \omega_1, ...\}$ be an infinite sequence of circles, such that $\omega_0$ has radius $\frac{1}{2}$ and each $\omega_k$ is internally tangent to $C_1$ and externally tangent to both $C_2$ and $\omega_{k+1}$. (The $\omega_k$'s are mutually distinct.) What is the radius of $\omega_{100}$?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Tiebreaker Round

**1** Daniel has an unlimited supply of tiles labeled "$2$" and "$n$" where $n$ is an integer. Find (with proof) all the values of $n$ that allow Daniel to fill an $8 \times 10$ grid with these tiles such that the sum of the values of the tiles in each row or column is divisible by $11$.

**2** Prove that if a triangle has integer side lengths and the area (in square units) equals the perimeter (in units), then the perimeter is not a prime number.

**3** Suppose $n$ points are uniformly chosen at random on the circumference of the unit circle and that they are then connected with line segments to form an $n$-gon. What is the probability that the origin is contained in the interior of this $n$-gon? Give your answer in terms of $n$, and consider only $n \geq 3$.

**4** A sequence of integers fsng is defined as follows: fix integers $a$, $b$, $c$, and $d$, then set $s_1 = a$, $s_2 = b$, and

$$s_n = cs_{n-1} + ds_{n-2}$$

for all $n \geq 3$. Create a second sequence $\{t_n\}$ by defining each $t_n$ to be the remainder when $s_n$ is divided by $2018$ (so we always have $0 \leq t_n \leq 2017$). Let $N = (2018^2)!$. Prove that $t_N = t_{2N}$ regardless of the choices of $a$, $b$, $c$, and $d$.

– Mixer Round

**Mixer Round** **p1.** Suppose $\frac{x}{y} = 0.\overline{ab}$ where $x$ and $y$ are relatively prime positive integers and $ab+a+b+1$ is a multiple of $12$. Find the sum of all possible values of $y$.

**p2.** Let $A$ be the set of points $\{(0,0),(2,0),(0,2),(2,2),(3,1),(1,3)\}$. How many distinct circles pass through at least three points in $A$?

**p3.** Jack and Jill need to bring pails of water home. The river is the $x$-axis, Jack is initially at the point $(-5,3)$, Jill is initially at the point $(6,1)$, and their home is at the point $(0,h)$ where $h>0$. If they take the shortest paths home given that each of them must make a stop at the river, they walk exactly the same total distance. What is $h$?

**p4.** What is the largest perfect square which is not a multiple of $10$ and which remains a perfect square if the ones and tens digits are replaced with zeroes?

**p5.** In convex polygon $P$, each internal angle measure (in degrees) is a distinct integer. What is the maximum possible number of sides $P$ could have?

**p6.** How many polynomials $p(x)$ of degree exactly $3$ with real coefficients satisfy

$$p(0), p(1), p(2), p(3) \in \{0, 1, 2\}?$$

**p7.** Six spheres, each with radius $4$, are resting on the ground. Their centers form a regular hexagon, and adjacent spheres are tangent. A seventh sphere, with radius $13$, rests on top of and is tangent to all six of these spheres. How high above the ground is the center of the seventh sphere?

**p8.** You have a paper square. You may fold it along any line of symmetry. (That is, the layers of paper must line up perfectly.) You then repeat this process using the folded piece of paper. If the direction of the folds does not matter, how many ways can you make exactly eight folds while following these rules?

**p9.** Quadrilateral $ABCD$ has $\overline{AB} = 40$, $\overline{CD} = 10$, $\overline{AD} = \overline{BC}$, $m\angle BAD = 20^o$, and $m\angle ABC = 70^o$. What is the area of quadrilateral $ABCD$?

**p10.** We say that a permutation $\sigma$ of the set $\{1, 2, ..., n\}$ preserves divisibilty if $\sigma(a)$ divides $\sigma(b)$ whenever $a$ divides $b$. How many permutations of $\{1, 2, ..., 40\}$ preserve divisibility? (A permutation of $\{1, 2, ..., n\}$ is a function $\sigma$ from $\{1, 2, ..., n\}$ to itself such that for any $b \in \{1, 2, ..., n\}$, there exists some $a \in \{1, 2, ..., n\}$ satisfying $\sigma(a) = b$.)

**p11.** In the diagram shown at right, how many ways are there to remove at least one edge so that some circle with an ”A” and some circle with a ”B” remain connected?
https://cdn.artofproblemsolving.com/attachments/8/7/fde209c63cc23f6d3482009cc6016c7cefc86
png

**p12.** Let $S$ be the set of the $125$ points in three-dimension space of the form $(x, y, z)$ where $x$, $y$, and $z$ are integers between $1$ and $5$, inclusive. A family of snakes lives at the point $(1, 1, 1)$, and one day they decide to move to the point $(5, 5, 5)$. Snakes may slither only in increments of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Given that at least one snake has slithered through each point of $S$ by the time the entire family has reached $(5, 5, 5)$, what is the smallest number of snakes that could be in the family?

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