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TST-1 TST-1

1 Find all integer values of x for which the value of the expression

$$x^2+6x+33$$

is a perfect square.

2 Let $ABCD$ be a square. Let E, Z be points on the sides AB, CD of the square respectively, such that $DE \parallel BZ$. Assume that the triangles $\triangle EAD, \triangle ZCB$ and the parallelogram $BEDZ$ have the same area.

If the distance between the parallel lines DE and BZ is equal to 1, determine the area of the square.

3 If x, y are real numbers with $x+y \geqslant 0$, determine the minimum value of the expression

$$K=x^5+y^5-x^4y-xy^4+x^2+4x+7$$

For which values of x, y does K take its minimum value?

4 Consider the digits $1,2,3,4,5,6,7$.
(a) Determine the number of seven-digit numbers with distinct digits that can be constructed using the digits above.
(b) If we place all of these seven-digit numbers in increasing order, find the seven-digit number which appears in the 2022^{th} position.

TST-2 TST-2

1 Determine all real numbers $x \in \mathbb{R}$ for which

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor = x-2022.$$

The notation $\lfloor z \rfloor$, for $z \in \mathbb{R}$, denotes the largest integer which is less than or equal to z. For example:

$$\lfloor 3.98 \rfloor = 3 \quad \text{and} \quad \lfloor 0.14 \rfloor = 0.$$

2 Determine all pairs of prime numbers (p,q) which satisfy the equation

$$p^3+q^3+1=p^2q^2$$

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3 Let ABC be an acute-angled triangle, and let D,E and K be the midpoints of its sides AB,AC and BC respectively. Let O be the circumcentre of triangle ABC, and let M be the foot of the perpendicular from A on the line BC. From the midpoint P of OM we draw a line parallel to AM, which meets the lines DE and OA at the points T and Z respectively. Prove that:

(a) the triangle DZE is isosceles
(b) the area of the triangle DZE is given by the formula

$$E_{DZE}=\frac{BC\cdot OK}{8}$$

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4 Let A be a subset of $\{1,2,3,\ldots,50\}$ with the property: for every $x,y\in A$ with $x\neq y$, it holds that

$$\left|\frac{1}{x}-\frac{1}{y}\right|>\frac{1}{1000}.$$

Determine the largest possible number of elements that the set A can have.

TST-3 TST-3

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1 Prove that for every natural number k, at least one of the integers

$$2k-1,\quad 5k-1\quad \text{and}\quad 13k-1$$

is not a perfect square.

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2 In a triangle ABC with $\widehat{A}=80^\circ$ and $\widehat{B}=60^\circ$, the internal angle bisector of $\widehat{C}$ meets the side AB at the point D. The parallel from D to the side AC, meets the side BC at the point E.

Find the measure of the angle $\angle EAB$.

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3 If a,b,c are positive real numbers with $abc=1$, prove that
(a)

$$2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)\geqslant\frac{9}{ab+bc+ca}$$

(b)

$$2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)\geqslant\frac{9}{a^2b+b^2c+c^2a}$$

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4 The numbers $1, 2, 3, \ldots, 10$ are written on the blackboard. In each step, Andrew chooses two numbers a, b which are written on the blackboard such that $a \geqslant 2b$, he erases them, and in their place writes the number $a - 2b$.

Find all numbers n, such that after a sequence of steps as above, at the end only the number n will remain on the blackboard.