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TST 1 TST 1

1 Find all pairs of real numbers (x, y) for which

$$x^2 + y^2 + xy = 133$$
$$x + y + \sqrt{xy} = 19$$

2 Let n, m be positive integers such that

$$n(4n + 1) = m(5m + 1)$$

(a) Show that the difference $n - m$ is a perfect square of a positive integer.
(b) Find a pair of positive integers (n, m) which satisfies the above relation.

Additional part (not asked in the TST): Find all such pairs (n, m).

3 Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$ and let (c) be its circumcircle with center O. Let M be the midpoint of BC. The line AM meets the circle (c) again at the point D. The circumcircle (c_1) of triangle $\triangle MDC$ intersects the line AC at the points C and I, and the circumcircle (c_2) of $\triangle AMI$ intersects the line AB at the points A and Z.

If N is the foot of the perpendicular from B on AC, and P is the second point of intersection of ZN with (c_2), prove that the quadrilateral with vertices the points N, P, I and M is a parallelogram.

4 Let m, n be positive integers with $m < n$ and consider an $n \times n$ board from which its upper left $m \times m$ part has been removed. An example of such board for $n = 5$ and $m = 2$ is shown below.

Determine for which pairs (m, n) this board can be tiled with 3×1 tiles. Each tile can be positioned either horizontally or vertically so that it covers exactly three squares of the board. The tiles should not overlap and should not cover squares outside of the board.

TST 2 TST 2

1 Find all pairs of integers (m, n) which satisfy the equation

$$(2n^2 + 5m - 5n - mn)^2 = m^3n$$

2 Determine for how many positive integers $n \in \{1, 2, \ldots, 2022\}$ it holds that 402 divides at least one of

$$n^2 - 1, n^3 - 1, n^4 - 1$$

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3 Let ABC be an obtuse-angled triangle with $\angle ABC > 90^\circ$, and let (c) be its circumcircle. The internal angle bisector of $\angle BAC$ meets again the circle (c) at the point E, and the line BC at the point D. The circle of diameter DE meets the circle (c) at the point H.

If the line HE meets the line BC at the point K, prove that:

(a) the points K, H, D and A are concyclic
(b) the line AH passes through the point of intersection of the tangents to the circle (c) at the points B and C.

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4 Let

$$M = \{1, 2, 3, \ldots, 2022\}$$

Determine the least positive integer k, such that for every k subsets of M with the cardinality of each subset equal to 3, there are two of these subsets with exactly one common element.
