Greece National Olympiad 2022

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1 Let ABC be a triangle such that $AB < AC < BC$. Let D, E be points on the segment BC such that $BD = BA$ and $CE = CA$. If K is the circumcenter of triangle ADE, F is the intersection of lines AD, KC and G is the intersection of lines AE, KB, then prove that the circumcircle of triangle KDE (let it be c_1), the circle with center the point F and radius FE (let it be c_2) and the circle with center G and radius GD (let it be c_3) concur on a point which lies on the line AK.

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2 Let $n > 4$ be a positive integer, which is divisible by 4. We denote by A_n the sum of the odd positive divisors of n. We also denote B_n the sum of the even positive divisors of n, excluding the number n itself. Find the least possible value of the expression

$$f(n) = B_n - 2A_n,$$

for all possible values of n, as well as for which positive integers n this minimum value is attained.

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3 The positive real numbers a, b, c, d satisfy the equality

$$a + bc + cd + db + \frac{1}{ab^2c^2d^2} = 18.$$

Find the maximum possible value of a.

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4 Let Q_n be the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \{0, 1, 2\}$, $i = 1, 2, \ldots, n$. A triple (x, y, z) (where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $z = (z_1, z_2, \ldots, z_n)$) of distinct elements of Q_n is called a *good* triple, if there exists at least one $i \in \{1, 2, \ldots, n\}$, for which $\{x_i, y_i, z_i\} = \{0, 1, 2\}$. A subset A of Q_n will be called a *good* subset, if any three elements of A form a *good* triple. Prove that every *good* subset of Q_n contains at most $2 \cdot \left(\frac{3}{2}\right)^n$ elements.
