

Utah Mathematical Olympiad 2016

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- 1 Ada and Otto are engaged in a battle of wits. In front of them is a figure with six dots, and nine sticks are placed between pairs of dots as shown below. The dots are labeled A, B, C, D, E, F. Ada begins the game by placing a pebble on the dot of her choice. Then, starting with Ada and alternating turns, each player picks a stick adjacent to the pebble, moves the pebble to the dot at the other end of the stick, and then removes the stick from the figure. The game ends when there are no sticks adjacent to the pebble. The player who moves last wins. A sample game is described below. If both players play optimally, who will win?

<https://cdn.artofproblemsolving.com/attachments/7/8/018aec96905b42b11e9d30b48ecf638779ea0.png>

Sample Game

1. Ada places the pebble at B.
2. Ada removes the stick BC, placing the pebble at C.
3. Otto removes the stick CD, placing the pebble at D.
4. Ada removes the stick DE, placing the pebble at E.
5. Otto removes the stick EA, placing the pebble at A.
6. Ada removes the stick AB and wins.

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- 2 Four fair six-sided dice are rolled. What is the probability that they can be divided into two pairs which sum to the same value? For example, a roll of $(1, 4, 6, 3)$ can be divided into $(1, 6)$ and $(4, 3)$, each of which sum to 7, but a roll of $(1, 1, 5, 2)$ cannot be divided into two pairs that sum to the same value.

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- 3 Can each positive integer $1, 2, 3, \dots$ be colored either red or blue, such that for all positive integers a, b, c, d (not necessarily distinct), if $a + b + c = d$ then a, b, c, d are not all the same color?

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- 4 Equiangular hexagon $ABCDEF$ has $AB = CD = EF$ and $AB > BC$. Segments AD and CF intersect at point X and segments BE and CF intersect at point Y . If quadrilateral $ABYX$ can have a circle inscribed inside of it (meaning there exists a circle that is tangent to all four sides of the quadrilateral), then find $\frac{AB}{FA}$.

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- 5 Let a_0, a_1, a_2, \dots be a sequence of integers (positive, negative, or zero) such that for all nonnegative integers n and k ,

$$a_{n+k}^2 - (2k + 1)a_n a_{n+k} + (k^2 + k)a_n^2 = k^2 - k.$$

Find all possible sequences (a_n) .

- 6 Find all positive integer pairs (u, m) such that $u + m^2$ is divisible by $um - 1$.
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