Utah Mathematical Olympiad 2018

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by parmenides51

1 (a) Show that you can draw 7 circles and 11 dots on a page that satisfy all of the following properties: • No two circles intersect. • No dot lies on a circle. • Each circle encloses a different number of total dots (which could be 0). • Each dot is enclosed by the same number of total circles.
(b) Prove that this is not possible with 5 circles and 5 dots.

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2 Let $P(x)$ be a cubic polynomial $(x-a)(x-b)(x-c)$, where a, b, and c are positive real numbers. Let $Q(x)$ be the polynomial with $Q(x) = (x-ab)(x-bc)(x-ca)$. If $P(x) = Q(x)$ for all x, then find the minimum possible value of $a+b+c$.

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3 Kim and Li play the following game. Kim places a 2×1 rectangle vertically somewhere on the 4×4 grid below. Then Li places a 1×2 rectangle on the grid horizontally so that it does not overlap with Kim's rectangle. These plays must be in-line with the grid, so no rectangle can partially cover more than two squares on the grid. Play repeats in this fashion until one of the players is unable to place any more rectangles. The last player to move wins.
If both players play optimally, determine, with proof, who wins.
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png

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4 Square $ABCD$ lies inside triangle XYZ such that B is on $\overline{XY}$, C is on $\overline{YZ}$, and D is on $\overline{ZX}$. Segment $\overline{XA}$ is also drawn. The result is four triangular regions ($\triangle XAB$, $\triangle YBC$, $\triangle ZCD$, and $\triangle XDA$), each adjacent to one of the four sides of the square. Prove that if these four triangular regions are each reflected across the adjacent side of the square, the resulting four reflected regions completely cover the square.
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5 Twelve labeled points are spaced equally on a circle, as shown below. How many ways are there to color 3 of the points red, 3 green, 3 blue, and 3 white such that for every two different colors, there exists a straight line that separates all the points of those two colors? (For example, there should be a line such that all the green points are on one side, and all the white points are on the other.)
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6 If p and q are distinct prime numbers, then determine all possible values of

$$gcd\left(p-1, \frac{q^p-1}{q-1}\right).$$

(For positive integers x and y, $gcd(x, y)$ denotes the greatest common divisor of x and y.)