Utah Mathematical Olympiad 2021

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1 Find all ordered triples of integers (x, b, c), such that b is prime, c is odd and positive, and $x^2 + bx + c = 0$.

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2 Three circles, C_1, C_2, C_3, are drawn in the plane such that each pair is externally tangent. Circle D is drawn externally tangent to all three, and circle E internally tangent to all three. If D and E have the same center, prove of disprove that C_1, C_2, and C_3 must have the same radius.

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3 To each point P in the plane, a real number $f(P)$ is assigned. Is it possible that for every equilateral triangle PQR in the plane, $f(P) + f(Q) + f(R)$ is equal to the perimeter of $\triangle PQR$?

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4 Farmer Georgia has a positive integer number number c cows (which have four legs), and zero ostriches, on her farm on day 1. On each day thereafter, she adds a combination of cows and ostriches to her farm, so that on each day $n \geq 2$, the number of animals on the farm is equal to exactly half the number of legs that were on the farm on day $n - 1$. For example, there are $4c$ legs on day 1, so there must be exactly $2c$ animals on day 2. She may never remove animals from the farm.
Let $P_c(n)$ be the number of possible sequences of ordered pairs $(c_1, o_1), (c_2, o_2), \ldots, (c_n, o_n)$ such that c_i, o_i are the number of cows and ostriches, respectively, on the farm on day i, where $(c_1, o_1) = (c, 0)$. For example, we have $P_1(2) = 2$, $P_1(3) = 5$, and $P_2(3) = 12$.
Find all positive integers c such that $P_c(2021)$ is a multiple of 3.

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5 Gog and Magog are playing a game with stones. Each player starts out with no stones, and they alternate taking turns. Gog goes first. On each turn, a player can either gain one new stone, or give at least one and no more than half of their stones to the other player. If a player has 20 or more stones, they lose.
Determine, with proof, whether Gog has a winning strategy, Magog has a winning strategy, or neither player has a winning strategy (the game goes on indefinitely).

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6 Prove that for all positive integer n, the number of divisors of $n!$ is a divisor of $(2n)!$.
