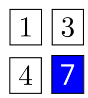
www.artofproblemsolving.com/community/c3001576
by parmenides51, wu2481632

– Individual Round

1 Charlotte is playing the hit new web number game, Primle. In this game, the objective is to guess a two-digit positive prime integer between 10 and 99, called the *Primle*. For each guess, a digit is highlighted blue if it is in the *Primle*, but not in the correct place. A digit is highlighted orange if it is in the *Primle* and is in the correct place. Finally, a digit is left unhighlighted if it is not in the *Primle*. If Charlotte guesses 13 and 47 and is left with the following game board, what is the *Primle*?



Proposed by Andrew Wu and Jason Wang

2 How many ways are there to fill in a 2×2 square grid with the numbers $1, 2, 3,$ and 4 such that the numbers in any two grid squares that share an edge have an absolute difference of at most 2?

Proposed by Andrew Wu

3 The **Collaptz function** is defined as

$$C(n) = \begin{cases} 3n - 1 & n \text{ odd}, \\ \frac{n}{2} & n \text{ even}. \end{cases}$$

We obtain the **Collaptz sequence** of a number by repeatedly applying the Collaptz function to that number. For example, the Collaptz sequence of 13 begins with $13, 38, 19, 56, 28, \cdots$ and so on. Find the sum of the three smallest positive integers n whose Collaptz sequences do not contain $1,$ or in other words, do not **collaptzse**.

Proposed by Andrew Wu and Jason Wang

4 Kara rolls a six-sided die, and if on that first roll she rolls an n, she rolls the die $n-1$ more times. She then computes that the product of all her rolls, including the first, is 8. How many distinct sequences of rolls could Kara have rolled?

Proposed by Andrew Wu

5 Cat and Claire are having a conversation about Cat's favorite number.

Cat says, "My favorite number is a two-digit positive integer with distinct nonzero digits, $\overline{AB}$, such that A and B are both factors of $\overline{AB}$."

Claire says, "I don't know your favorite number yet, but I do know that among four of the numbers that might be your favorite number, you could start with any one of them, add a second, subtract a third, and get the fourth!"

Cat says, "That's cool, and my favorite number is among those four numbers! Also, the square of my number is the product of two of the other numbers among the four you mentioned!"

Claire says, "Now I know your favorite number!"

What is Cat's favorite number?

Proposed by Andrew Wu

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6 Carissa is crossing a very, very, very wide street, and did not properly check both ways before doing so. (Don't be like Carissa!) She initially begins walking at 2 feet per second. Suddenly, she hears a car approaching, and begins running, eventually making it safely to the other side, half a minute after she began crossing. Given that Carissa always runs n times as fast as she walks and that she spent n times as much time running as she did walking, and given that the street is 260 feet wide, find Carissa's running speed, in feet per second.

Proposed by Andrew Wu

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7 Given that six-digit positive integer $\overline{ABCDEF}$ has distinct digits A, B, C, D, E, F between 1 and 8, inclusive, and that it is divisible by 99, find the maximum possible value of $\overline{ABCDEF}$.

Proposed by Andrew Milas

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8 Triangle ABC has sidelengths $AB = 1$, $BC = \sqrt{3}$, and $AC = 2$. Points D, E, and F are chosen on AB, BC, and AC respectively, such that $\angle EDF = \angle DFA = 90^\circ$. Given that the maximum possible value of $[DEF]^2$ can be expressed as $\frac{a}{b}$ for positive integers a, b with $\gcd(a, b) = 1$, find $a + b$. (Here $[DEF]$ denotes the area of triangle DEF.)

Proposed by Vismay Sharan

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9 Suppose that $P(x)$ is a monic quadratic polynomial satisfying $aP(a) = 20P(20) = 22P(22)$ for some integer $a \neq 20, 22$. Find the minimum possible positive value of $P(0)$.

Proposed by Andrew Wu

(Note: wording changed from original to specify that $a \neq 20, 22$.)

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10 How many ways are there to choose distinct positive integers a, b, c, d dividing 15^6 such that none of a, b, c, or d divide each other? (Order does not matter.)

Proposed by Miles Yamner and Andrew Wu

(Note: wording changed from original to clarify)

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11 Georgina calls a 992-element subset A of the set $S = \{1, 2, 3, \dots, 1984\}$ a **halfthink set** if

- the sum of the elements in A is equal to half of the sum of the elements in S, and
- exactly one pair of elements in A differs by 1.

She notices that for some values of n, with n a positive integer between 1 and 1983, inclusive, there are no halfthink sets containing both n and $n+1$. Find the last three digits of the product of all possible values of n.

Proposed by Andrew Wu and Jason Wang

(Note: wording changed from original to specify what n can be.)

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12 Let ABC be a triangle with $AB = 5$, $BC = 7$, and $CA = 8$, and let D be a point on arc $\widehat{BC}$ of its circumcircle Ω. Suppose that the angle bisectors of $\angle ADB$ and $\angle ADC$ meet AB and AC at E and F, respectively, and that EF and BC meet at G. Line GD meets Ω at T. If the maximum possible value of AT^2 can be expressed as $\frac{a}{b}$ for positive integers a, b with $\gcd(a, b) = 1$, find $a + b$.

Proposed by Andrew Wu

Tiebreaker **p1.** Suppose that x and y are positive real numbers such that $\log_2 x = \log_x y = \log_y 256$. Find xy.

p2. Let the roots of $x^2 + 7x + 11$ be r and s. If f(x) is the monic polynomial with roots $rs + r + s$ and $r^2 + s^2$, what is $f(3)$?

p3. Call a positive three digit integer $\overline{ABC}$ fancy if $\overline{ABC} = (\overline{AB})^2 - 11 \cdot \overline{C}$. Find the sum of all fancy integers.

p4. In triangle ABC, points D and E are on line segments BC and AC, respectively, such that AD and BE intersect at H. Suppose that $AC = 12$, $BC = 30$, and $EC = 6$. Triangle BEC has area 45 and triangle ADC has area 72, and lines CH and AB meet at F. If BF^2 can be expressed as $\frac{a-b\sqrt{c}}{d}$ for positive integers a, b, c, d with c squarefree and $gcd(a, b, d) = 1$, then find $a+b+c+d$.

p5. Find the minimum possible integer y such that $y > 100$ and there exists a positive integer x such that $x^2 + 18x + y$ is a perfect fourth power.

p6. Let $ABCD$ be a quadrilateral such that $AB = 2$, $CD = 4$, $BC = AD$, and $\angle ADC + \angle BCD =$

120^o. If the sum of the maximum and minimum possible areas of quadrilateral $ABCD$ can be expressed as $a\sqrt{b}$ for positive integers a, b with b squarefree, then find $a + b$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Mathathon Round

R1 **p1** How many two-digit positive integers with distinct digits satisfy the conditions that
1) neither digit is 0, and
2) the units digit is a multiple of the tens digit?

p2 Mirabel has 47 candies to pass out to a class with n students, where $10 \le n < 20$. After distributing the candy as evenly as possible, she has some candies left over. Find the smallest integer k such that Mirabel could have had k leftover candies.

p3 Callie picks two distinct numbers from $\{1, 2, 3, 4, 5\}$ at random. The probability that the sum of the numbers she picked is greater than the sum of the numbers she didn't pick is p. p can be expressed as $\frac{a}{b}$ for positive integers a, b with $gcd(a, b) = 1$. Find $a + b$.

R2 **p4** Define the sequence a_n as follows:
1) $a_1 = -1$, and
2) for all $n \ge 2$, $a_n = 1 + 2 + ... + n - (n + 1)$.
For example, $a_3 = 1+2+3-4 = 2$. Find the largest possible value of k such that $a_k+a_{k+1} = a_{k+2}$.

p5 The taxicab distance between two points (a, b) and (c, d) on the coordinate plane is $|c - a| + |d - b|$. Given that the taxicab distance between points A and B is 8 and that the length of AB is k, find the minimum possible value of k^2.

p6 For any two-digit positive integer $\overline{AB}$, let $f(\overline{AB}) = \overline{AB} - A \cdot B$, or in other words, the result of subtracting the product of its digits from the integer itself. For example, $f(\overline{72}) = 72 - 7 \cdot 2 = 58$. Find the maximum possible n such that there exist distinct two-digit integers $\overline{XY}$ and $\overline{WZ}$ such that $f(\overline{XY}) = f(\overline{WZ}) = n$.

R3 **p7** Cindy cuts regular hexagon $ABCDEF$ out of a sheet of paper. She folds B over AC, resulting in a pentagon. Then, she folds A over CF, resulting in a quadrilateral. The area of $ABCDEF$ is k times the area of the resulting folded shape. Find k.

p8 Call a sequence $\{a_n\} = a_1, a_2, a_3, ...$ of positive integers *Fib-o'nacci* if it satisfies $a_n = a_{n-1} +$

a_{n-2} for all $n \geq 3$. Suppose that m is the largest even positive integer such that exactly one *Fib-o'nacci* sequence satisfies $a_5 = m$, and suppose that n is the largest odd positive integer such that exactly one *Fib-o'nacci* sequence satisfies $a_5 = n$. Find mn.

p9 Compute the number of ways there are to pick three non-empty subsets A, B, and C of $\{1, 2, 3, 4, 5, 6\}$, such that $|A| = |B| = |C|$ and the following property holds:

$$A \cap B \cap C = A \cap B = B \cap C = C \cap A.$$

R4 **p10** Kathy has two positive real numbers, a and b. She mistakenly writes

$$\log(a + b) = \log(a) + \log(b),$$

but miraculously, she finds that for her combination of a and b, the equality holds. If $a = 2022b$, then $b = \frac{p}{q}$, for positive integers p, q where $gcd(p, q) = 1$. Find $p + q$.

p11 Points X and Y lie on sides AB and BC of triangle ABC, respectively. Ray $\overrightarrow{XY}$ is extended to point Z such that A, C, and Z are collinear, in that order. If triangleABZ is isosceles and triangle CYZ is equilateral, then the possible values of $\angle ZXB$ lie in the interval $I = (a^o, b^o)$, such that $0 \leq a, b \leq 360$ and such that a is as large as possible and b is as small as possible. Find $a + b$.

p12 Let $S = \{(a, b)|0 \leq a, b \leq 3, a \text{ and } b \text{ are integers } \}$. In other words, S is the set of points in the coordinate plane with integer coordinates between 0 and 3, inclusive. Prair selects four distinct points in S, for each selected point, she draws lines with slope 1 and slope -1 passing through that point. Given that each point in S lies on at least one line Prair drew, how many ways could she have selected those four points?

R5 **p13** Let $ABCD$ be a square. Points E and F lie outside of $ABCD$ such that ABE and CBF are equilateral triangles. If G is the centroid of triangle DEF, then find $\angle AGC$, in degrees.

p14 The silent reading $s(n)$ of a positive integer n is the number obtained by dropping the zeros not at the end of the number. For example, $s(1070030) = 1730$. Find the largest $n < 10000$ such that $s(n)$ divides n and $n \neq s(n)$.

p15 Let $ABCDEFGH$ be a regular octagon with side length 12. There exists a region R inside the octagon such that for each point X in R, exactly three of the rays AX, BX, CX, DX, GX, and HX intersect segment EF. If the area of region R can be expressed as $a - b\sqrt{c}$ for positive integers a, b, c with c squarefree, find $a + b + c$.

R6 **p16** Madelyn is being paid $\$50$/hour to find useful *Non-Functional Trios*, where a Non-Functional Trio is defined as an ordered triple of distinct real numbers (a, b, c), and a Non- Functional Trio is *useful* if (a, b), (b, c), and (c, a) are collinear in the Cartesian plane. Currently, she's working on the case $a + b + c = 2022$. Find the number of useful Non-Functional Trios (a, b, c) such that $a + b + c = 2022$.

p17 Let $p(x) = x^2 - k$, where k is an integer strictly less than 250. Find the largest possible value of k such that there exist distinct integers a, b with $p(a) = b$ and $p(b) = a$.

p18 Let ABC be a triangle with orthocenter H and circumcircle Γ such that $AB = 13$, $BC = 14$, and $CA = 15$. BH and CH meet Γ again at points D and E, respectively, and DE meets AB and AC at F and G, respectively. The circumcircles of triangles ABG and ACF meet BC again at points P and Q. If PQ can be expressed as $\frac{a}{b}$ for positive integers a, b with $gcd(a, b) = 1$, find $a + b$.

Mixer Round **p1.** Find the smallest positive integer N such that $2N - 1$ and $2N + 1$ are both composite.

p2. Compute the number of ordered pairs of integers (a, b) with $1 \le a, b \le 5$ such that $ab - a - b$ is prime.

p3. Given a semicircle Ω with diameter AB, point C is chosen on Ω such that $\angle CAB = 60^o$. Point D lies on ray BA such that DC is tangent to Ω. Find $\left(\frac{BD}{BC}\right)^2$.

p4. Let the roots of $x^2 + 7x + 11$ be r and s. If $f(x)$ is the monic polynomial with roots $rs + r + s$ and $r^2 + s^2$, what is $f(3)$?

p5. Regular hexagon $ABCDEF$ has side length 3. Circle ω is drawn with AC as its diameter. BC is extended to intersect ω at point G. If the area of triangle BEG can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with b squarefree and $gcd(a, c) = 1$, find $a + b + c$.

p6. Suppose that x and y are positive real numbers such that $\log_2 x = \log_x y = \log_y 256$. Find xy.

p7. Call a positive three digit integer $\overline{ABC}$ fancy if $\overline{ABC} = (\overline{AB})^2 - 11 \cdot \overline{C}$. Find the sum of all fancy integers.

p8. Let $\triangle ABC$ be an equilateral triangle. Isosceles triangles $\triangle DBC$, $\triangle ECA$, and $\triangle FAB$, not overlapping $\triangle ABC$, are constructed such that each has area seven times the area of $\triangle ABC$. Compute the ratio of the area of $\triangle DEF$ to the area of $\triangle ABC$.

p9. Consider the sequence of polynomials an(x) with $a_0(x) = 0$, $a_1(x) = 1$, and $a_n(x) = a_{n-1}(x) + xa_{n-2}(x)$ for all $n \geq 2$. Suppose that $p_k = a_k(-1) \cdot a_k(1)$ for all nonnegative integers k. Find the number of positive integers k between 10 and 50, inclusive, such that $p_{k-2} + p_{k-1} = p_{k+1} - p_{k+2}$.

p10. In triangle ABC, point D and E are on line segments BC and AC, respectively, such that AD and BE intersect at H. Suppose that $AC = 12$, $BC = 30$, and $EC = 6$. Triangle BEC has area 45 and triangle ADC has area 72, and lines CH and AB meet at F. If BF^2 can be expressed as $\frac{a-b\sqrt{c}}{d}$ for positive integers a, b, c, d with c squarefree and $gcd(a, b, d) = 1$, then find $a+b+c+d$.

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