Exeter Math Club Competition 2011

www.artofproblemsolving.com/community/c3003270

by parmenides51

– Team Round

– **p1.** Velociraptor A is located at $x = 10$ on the number line and runs at 4 units per second. Velociraptor B is located at $x = -10$ on the number line and runs at 3 units per second. If the velociraptors run towards each other, at what point do they meet?

p2. Let n be a positive integer. There are n non-overlapping circles in a plane with radii $1, 2, ..., n$. The total area that they enclose is at least 100. Find the minimum possible value of n.

p3. How many integers between 1 and 50, inclusive, are divisible by 4 but not 6?

p4. Let $a \star b = 1 + \frac{b}{a}$. Evaluate $((((((1 \star 1) \star 1) \star 1) \star 1) \star 1) \star 1) \star 1$.

p5. In acute triangle ABC, D and E are points inside triangle ABC such that $DE \parallel BC$, B is closer to D than it is to E, $\angle AED = 80^o$, $\angle ABD = 10^o$, and $\angle CBD = 40^o$. Find the measure of $\angle BAE$, in degrees.

p6. Al is at $(0, 0)$. He wants to get to $(4, 4)$, but there is a building in the shape of a square with vertices at $(1, 1)$, $(1, 2)$, $(2, 2)$, and $(2, 1)$. Al cannot walk inside the building. If Al is not restricted to staying on grid lines, what is the shortest distance he can walk to get to his destination?

p7. Point $A = (1, 211)$ and point $B = (b, 2011)$ for some integer b. For how many values of b is the slope of AB an integer?

p8. A palindrome is a number that reads the same forwards and backwards. For example, 1, 11 and 141 are all palindromes. How many palindromes between 1 and 1000 are divisible by 11?

p9. Suppose x, y, z are real numbers that satisfy:

$$x + y - z = 5$$

$$y + z - x = 7$$

$$z + x - y = 9$$

Find $x^2 + y^2 + z^2$.

p10. In triangle ABC, $AB = 3$ and $AC = 4$. The bisector of angle A meets BC at D. The line through D perpendicular to AD intersects lines AB and AC at F and E, respectively. Compute $EC - FB$. (See the following diagram.)
https://cdn.artofproblemsolving.com/attachments/2/7/e26fbaeb7d1f39cb8d5611c6a466add881ba0
png

p11. Bob has a six-sided die with a number written on each face such that the sums of the numbers written on each pair of opposite faces are equal to each other. Suppose that the numbers 109, 131, and 135 are written on three faces which share a corner. Determine the maximum possible sum of the numbers on the three remaining faces, given that all three are positive primes less than 200.

p12. Let d be a number chosen at random from the set $\{142, 143, ..., 198\}$. What is the probability that the area of a rectangle with perimeter 400 and diagonal length d is an integer?

p13. There are 3 congruent circles such that each circle passes through the centers of the other two. Suppose that A, B, and C are points on the circles such that each circle has exactly one of A, B, or C on it and triangle ABC is equilateral. Find the ratio of the maximum possible area of ABC to the minimum possible area of ABC. (See the following diagram.)
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p14. Let k and m be constants such that for all triples (a, b, c) of positive real numbers,

$$\sqrt{\frac{4}{a^2} + \frac{36}{b^2} + \frac{9}{c^2} + \frac{k}{ab}} = \left|\frac{2}{a} + \frac{6}{b} + \frac{3}{c}\right|$$

if and only if $am^2 + bm + c = 0$. Find k.

p15. A bored student named Abraham is writing n numbers $a_1, a_2, ..., a_n$. The value of each number is either $1, 2$, or 3; that is, a_i is $1, 2$ or 3 for $1 \le i \le n$. Abraham notices that the ordered triples

$$(a_1, a_2, a_3), (a_2, a_3, a_4), ..., (a_{n-2}, a_{n-1}, a_n), (a_{n-1}, a_n, a_1), (a_n, a_1, a_2)$$

are distinct from each other. What is the maximum possible value of n? Give the answer n, along with an example of such a sequence. Write your answer as an ordered pair. (For example, if the answer were 5, you might write $(5, 12311)$.)

PS. You had better use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Guts Round

– Round 1

p1. In order to make good salad dressing, Bob needs a 0.9% salt solution. If soy sauce is 15% salt, how much water, in mL, does Bob need to add to 3 mL of pure soy sauce in order to have a good salad dressing?

p2. Alex the Geologist is buying a canteen before he ventures into the desert. The original cost of a canteen is $\$20$, but Alex has two coupons. One coupon is $\$3$ off and the other is 10% off the entire remaining cost. Alex can use the coupons in any order. What is the least amount of money he could pay for the canteen?

p3. Steve and Yooni have six distinct teddy bears to split between them, including exactly 1 blue teddy bear and 1 green teddy bear. How many ways are there for the two to divide the teddy bears, if Steve gets the blue teddy bear and Yooni gets the green teddy bear? (The two do not necessarily have to get the same number of teddy bears, but each teddy bear must go to a person.)

Round 2

p4. In the currency of Mathamania, 5 wampas are equal to 3 kabobs and 10 kabobs are equal to 2 jambas. How many jambas are equal to twenty-five wampas?

p5. A sphere has a volume of 81π. A new sphere with the same center is constructed with a radius that is $\frac{1}{3}$ the radius of the original sphere. Find the volume, in terms of π, of the region between the two spheres.

p6. A frog is located at the origin. It makes four hops, each of which moves it either 1 unit to the right or 1 unit to the left. If it also ends at the origin, how many 4-hop paths can it take?

Round 3

p7. Nick multiplies two consecutive positive integers to get $4^5 - 2^5$. What is the smaller of the two numbers?

p8. In rectangle $ABCD$, E is a point on segment CD such that $\angle EBC = 30^o$ and $\angle AEB = 80^o$. Find $\angle EAB$, in degrees.

p9. Mary's secret garden contains clones of Homer Simpson and WALL-E. A WALL-E clone has 4 legs. Meanwhile, Homer Simpson clones are human and therefore have 2 legs each. A Homer Simpson clone always has 5 donuts, while a WALL-E clone has 2. In Mary's secret garden, there are 184 donuts and 128 legs. How many WALL-E clones are there?

Round 4

p10. Including Richie, there are 6 students in a math club. Each day, Richie hangs out with a different group of club mates, each of whom gives him a dollar when he hangs out with them. How many dollars will Richie have by the time he has hung out with every possible group of club mates?

p11. There are seven boxes in a line: three empty, three holding $\$10$ each, and one holding the jackpot of $\$1,000,000$. From the left to the right, the boxes are numbered $1, 2, 3, 4, 5, 6$ and 7, in that order.
You are told the following: • No two adjacent boxes hold the same contents. • Box 4 is empty. • There is one more $\$10$ prize to the right of the jackpot than there is to the left.
Which box holds the jackpot?

p12. Let a and b be real numbers such that $a + b = 8$. Let c be the minimum possible value of $x^2 + ax + b$ over all real numbers x. Find the maximum possible value of c over all such a and b.

Round 5

p13. Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD, and P be a point on BM such that $BP = BC$. Find the area of $ABPD$.

p14. The number 19 has the following properties: • It is a 2-digit positive integer. • It is the two leading digits of a 4-digit perfect square, because $1936 = 44^2$.
How many numbers, including 19, satisfy these two conditions?

p15. In a 3×3 grid, each unit square is colored either black or white. A coloring is considered "nice" if there is at most one white square in each row or column. What is the total number of nice colorings? Rotations and reflections of a coloring are considered distinct. (For example, in

the three squares shown below, only the rightmost one has a nice coloring.
https://cdn.artofproblemsolving.com/attachments/e/4/e6932c822bec77aa0b07c98d1789e58416b91
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PS. You should use hide for answers. Rest rounds have been posted here (https://artofproblemsolving.com/community/c4h2786958p24498425). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 6

p16. Let $a_1, a_2, ..., a_{2011}$ be a sequence of numbers such that $a_1 = 2011$ and $a_1 + a_2 + ... + a_n = n^2 \cdot a_n$ for $n = 1, 2, ...2011$. (That is, $a_1 = 1^2 \cdot a_1$, $a_1 + a_2 = 2^2 \cdot a_2$, ...) Compute a_{2011}.

p17. Three rectangles, with dimensions 3×5, 4×2, and 6×4, are each divided into unit squares which are alternately colored black and white like a checkerboard. Each rectangle is cut along one of its diagonals into two triangles. For each triangle, let m be the total black area and n the total white area. Find the maximum value of $|m - n|$ for the 6 triangles.

p18. In triangle ABC, $\angle BAC = 90^o$, and the length of segment AB is 2011. Let M be the midpoint of BC and D the midpoint of AM. Let E be the point on segment AB such that $EM \parallel CD$. What is the length of segment BE?

Round 7

p19. How many integers from 1 to 100, inclusive, can be expressed as the difference of two perfect squares? (For example, $3 = 2^2 - 1^2$).

p20. In triangle ABC, $\angle ABC = 45$ and $\angle ACB = 60^o$. Let P and Q be points on segment BC, F a point on segment AB, and E a point on segment AC such that $FQ \parallel AC$ and $EP \parallel AB$. Let D be the foot of the altitude from A to BC. The lines AD, FQ, and PE form a triangle. Find the positive difference, in degrees, between the largest and smallest angles of this triangle.

p21. For real number x, $\lceil x \rceil$ is equal to the smallest integer larger than or equal to x. For example, $\lceil 3 \rceil = 3$ and $\lceil 2.5 \rceil = 3$. Let $f(n)$ be a function such that $f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$ for every integer n greater than 1. If $f(1) = 1$, find the maximum value of $f(k) - k$, where k is a positive integer less than or equal to 2011.

Round 8

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

p22. Let W be the answer to problem 24 in this guts round. Let $f(a) = \frac{1}{1-\frac{1}{1-\frac{1}{a}}}$. Determine$|f(2)+ ... + f(W)|$.

p23. Let X be the answer to problem 22 in this guts round. How many odd perfect squares are less than $8X$?

p24. Let Y be the answer to problem 23 in this guts round. What is the maximum number of points of intersections of two regular $(Y - 5)$-sided polygons, if no side of the first polygon is parallel to any side of the second polygon?

<u>Round 9</u>

p25. Cross country skiers $s_1, s_2, s_3, ..., s_7$ start a race one by one in that order. While each skier skis at a constant pace, the skiers do not all ski at the same rate. In the course of the race, each skier either overtakes another skier or is overtaken by another skier exactly two times. Find all the possible orders in which they can finish. Write each possible finish as an ordered septuplet (a, b, c, d, e, f, g) where a, b, c, d, e, f, g are the numbers $1 - 7$ in some order. (So a finishes first, b finishes second, etc.)

p26. Archie the Alchemist is making a list of all the elements in the world, and the proportion of earth, air, fire, and water needed to produce each. He writes the proportions in the form E:A:F:W. If each of the letters represents a whole number from 0 to 4, inclusive, how many different elements can Archie list? Note that if Archie lists wood as $2 : 0 : 1 : 2$, then $4 : 0 : 2 : 4$ would also produce wood. In addition, $0 : 0 : 0 : 0$ does not produce an element.

p27. Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD, and P be the point on BM such that $DP = DA$. Find the area of quadrilateral $ABPD$.

<u>Round 10</u>

p28. David the farmer has an infinitely large grass-covered field which contains a straight wall. He ties his cow to the wall with a rope of integer length. The point where David ties his rope to the wall divides the wall into two parts of length a and b, where $a > b$ and both are integers. The rope is shorter than the wall but is longer than a. Suppose that the cow can reach grass covering an area of $\frac{165\pi}{2}$. Find the ratio $\frac{a}{b}$. You may assume that the wall has 0 width.

p29. Let S be the number of ordered quintuples (a, b, x, y, n) of positive integers such that

$$\frac{a}{x}+\frac{b}{y}=\frac{1}{n}$$

$$abn = 2011^{2011}$$

Compute the remainder when S is divided by 2012.

p30. Let n be a positive integer. An $n \times n$ square grid is formed by n^2 unit squares. Each unit square is then colored either red or blue such that each row or column has exactly 10 blue squares. A move consists of choosing a row or a column, and recolor each unit square in the chosen row or column – if it is red, we recolor it blue, and if it is blue, we recolor it red. Suppose that it is possible to obtain fewer than $10n$ blue squares after a sequence of finite number of moves. Find the maximum possible value of n.

PS. You should use hide for answers. First rounds have been posted here (`https://artofproblemsolving.com/community/c4h2786905p24497746`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Accuracy

– **p1.** What is the maximum number of points of intersection between a square and a triangle, assuming that no side of the triangle is parallel to any side of the square?

p2. Two angles of an isosceles triangle measure 80^o and x^o. What is the sum of all the possible values of x?

p3. Let p and q be prime numbers such that $p + q$ and p + $7q$ are both perfect squares. Find the value of pq.

p4. Anna, Betty, Carly, and Danielle are four pit bulls, each of which is either wearing or not wearing lipstick. The following three facts are true:
(1) Anna is wearing lipstick if Betty is wearing lipstick.
(2) Betty is wearing lipstick only if Carly is also wearing lipstick.
(3) Carly is wearing lipstick if and only if Danielle is wearing lipstick

The following five statements are each assigned a certain number of points:

(a) Danielle is wearing lipstick if and only if Carly is wearing lipstick. (This statement is assigned 1 point.)
(b) If Anna is wearing lipstick, then Betty is wearing lipstick. (This statement is assigned 6 points.)
(c) If Betty is wearing lipstick, then both Anna and Danielle must be wearing lipstick. (This statement is assigned 10 points.)
(d) If Danielle is wearing lipstick, then Anna is wearing lipstick. (This statement is assigned 12 points.)
(e) If Betty is wearing lipstick, then Danielle is wearing lipstick. (This statement is assigned 14 points.)
What is the sum of the points assigned to the statements that must be true? (For example, if only statements (a) and (d) are true, then the answer would be $1 + 12 = 13$.)

p5. Let $f(x)$ and $g(x)$ be functions such that $f(x) = 4x+3$ and $g(x) = \frac{x+1}{4}$. Evaluate $g(f(g(f(42))))$.

p6. Let A, B, C, and D be consecutive vertices of a regular polygon. If $\angle ACD = 120^o$, how many sides does the polygon have?

p7. Fred and George have a fair 8-sided die with the numbers $0, 1, 2, 9, 2, 0, 1, 1$ written on the sides. If Fred and George each roll the die once, what is the probability that Fred rolls a larger number than George?

p8. Find the smallest positive integer t such that $(23t)^3 - (20t)^3 - (3t)^3$ is a perfect square.

p9. In triangle ABC, $AC = 8$ and $AC < AB$. Point D lies on side BC with $\angle BAD = \angle CAD$. Let M be the midpoint of BC. The line passing through M parallel to AD intersects lines AB and AC at F and E, respectively. If $EF = \sqrt{2}$ and $AF = 1$, what is the length of segment BC? (See the following diagram.)
`https://cdn.artofproblemsolving.com/attachments/2/3/4b5dd0ae28b09f5289fb0e6c72c7cbf421d02`
`png`

p10. There are 2011 evenly spaced points marked on a circular table. Three segments are randomly drawn between pairs of these points such that no two segments share an endpoint on the circle. What is the probability that each of these segments intersects the other two?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Speed

– *20 problems for 20 minutes.*

p1. Euclid eats $\frac{1}{7}$ of a pie in 7 seconds. Euler eats $\frac{1}{5}$ of an identical pie in 10 seconds. Who eats faster?

p2. Given that $\pi = 3.1415926...$, compute the circumference of a circle of radius 1. Express your answer as a decimal rounded to the nearest hundred thousandth (i.e. 1.234562 and 1.234567 would be rounded to 1.23456 and 1.23457, respectively).

p3. Alice bikes to Wonderland, which is 6 miles from her house. Her bicycle has two wheels, and she also keeps a spare tire with her. If each of the three tires must be used for the same number of miles, for how many miles will each tire be used?

p4. Simplify $\frac{2010 \cdot 2010}{2011}$ to a mixed number. (For example, $2\frac{1}{2}$ is a mixed number while $\frac{5}{2}$ and 2.5 are not.)

p5. There are currently 175 problems submitted for EMC^2. Chris has submitted 51 of them. If nobody else submits any more problems, how many more problems must Chris submit so that he has submitted $\frac{1}{3}$ of the problems?

p6. As shown in the diagram below, points D and L are located on segment AK, with D between A and L, such that $\frac{AD}{DK} = \frac{1}{3}$ and $\frac{DL}{LK} = \frac{5}{9}$. What is $\frac{DL}{AK}$?
`https://cdn.artofproblemsolving.com/attachments/9/a/3f92bd33ffbe52a735158f7ebca79c4c360d3`
`png`

p7. Find the number of possible ways to order the letters G, G, e, e, e such that two neighboring letters are never G and e in that order.

p8. Find the number of odd composite integers between 0 and 50.

p9. Bob tries to remember his 2-digit extension number. He knows that the number is divisible by 5 and that the first digit is odd. How many possibilities are there for this number?

p10. Al walks 1 mile due north, then 2 miles due east, then 3 miles due south, and then 4 miles due west. How far, in miles, is he from his starting position? (Assume that the Earth is flat.)

p11. When n is a positive integer, $n!$ denotes the product of the first n positive integers; that is, $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n$. Given that $7! = 5040$, compute $8! + 9! + 10!$.

p12. Sam's phone company charges him a per-minute charge as well as a connection fee (which is the same for every call) every time he makes a phone call. If Sam was charged $\$4.88$ for an 11-minute call and $\$6.00$ for a 19-minute call, how much would he be charged for a 15-minute call?

p13. For a positive integer n, let s_n be the sum of the n smallest primes. Find the least n such that s_n is a perfect square (the square of an integer).

p14. Find the remainder when 2011^{2011} is divided by 7.

p15. Let a, b, c, and d be 4 positive integers, each of which is less than 10, and let e be their least common multiple. Find the maximum possible value of e.

p16. Evaluate $100 - 1 + 99 - 2 + 98 - 3 + ... + 52 - 49 + 51 - 50$.

p17. There are 30 basketball teams in the Phillips Exeter Dorm Basketball League. In how ways can 4 teams be chosen for a tournament if the two teams Soule Internationals and Abbot United cannot be chosen at the same time?

p18. The numbers $1, 2, 3, 4, 5, 6$ are randomly written around a circle. What is the probability that there are four neighboring numbers such that the sum of the middle two numbers is less than the sum of the other two?

p19. What is the largest positive 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$?

p20. Rhombus $ABCD$ has vertices $A = (-12, -4)$, $B = (6, b)$, $C = (c, -4)$ and $D = (d, -28)$, where b, c, and d are integers. Find a constant m such that the line y = mx divides the rhombus into two regions of equal area.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).