**Exeter Math Club Competition 2012**

www.artofproblemsolving.com/community/c3003271

by parmenides51

– Team Round

– **p1.** The longest diagonal of a regular hexagon is 12 inches long. What is the area of the hexagon, in square inches?

**p2.** When Al and Bob play a game, either Al wins, Bob wins, or they tie. The probability that Al does not win is $\frac23$ , and the probability that Bob does not win is $\frac34$ . What is the probability that they tie?

**p3.** Find the sum of the $a + b$ values over all pairs of integers $(a, b)$ such that $1 \le a < b \le 10$. That is, compute the sum

$$(1 + 2) + (1 + 3) + (1 + 4) + ... + (2 + 3) + (2 + 4) + ... + (9 + 10).$$

**p4.** A $3 \times 11$ cm rectangular box has one vertex at the origin, and the other vertices are above the $x$-axis. Its sides lie on the lines $y = x$ and $y = -x$. What is the $y$-coordinate of the highest point on the box, in centimeters?

**p5.** Six blocks are stacked on top of each other to create a pyramid, as shown below. Carl removes blocks one at a time from the pyramid, until all the blocks have been removed. He never removes a block until all the blocks that rest on top of it have been removed. In how many different orders can Carl remove the blocks?

`https://cdn.artofproblemsolving.com/attachments/b/e/9694d92eeb70b4066b1717fedfbfc601631ce`
`png`

**p6.** Suppose that a right triangle has sides of lengths $\sqrt{a+b\sqrt3},\sqrt{3+2\sqrt3}$, and $\sqrt{4+5\sqrt3}$, where $a, b$ are positive integers. Find all possible ordered pairs $(a, b)$.

**p7.** Farmer Chong Gu glues together $4$ equilateral triangles of side length $1$ such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

**p8.** Compute the number of ordered pairs $(a, b)$ of positive integers less than or equal to $100$, such that a $b - 1$ is a multiple of $4$.

**p9.** In triangle $ABC$, $\angle C = 90^o$. Point $P$ lies on segment $BC$ and is not $B$ or $C$. Point $I$ lies on segment $AP$. If $\angle BIP = \angle PBI = \angle CAB = m^o$ for some positive integer $m$, find the sum of all possible values of $m$.

**p10.** Bob has $2$ identical red coins and $2$ identical blue coins, as well as $4$ distinguishable buckets. He places some, but not necessarily all, of the coins into the buckets such that no bucket contains two coins of the same color, and at least one bucket is not empty. In how many ways can he do this?

**p11.** Albert takes a $4 \times 4$ checkerboard and paints all the squares white. Afterward, he wants to paint some of the square black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible configuration in which this holds. (Note: the answer sheet contains a $4 \times 4$ grid.)

**p12.** Let $S$ be the set of points $(x, y)$ with $0 \le x \le 5$, $0 \le y \le 5$ where $x$ and $y$ are integers. Let $T$ be the set of all points in the plane that are the midpoints of two distinct points in $S$. Let $U$ be the set of all points in the plane that are the midpoints of two distinct points in $T$. How many distinct points are in $U$? (Note: The points in $T$ and $U$ do not necessarily have integer coordinates.)

**p13.** In how many ways can one express $6036$ as the sum of at least two (not necessarily positive) consecutive integers?

**p14.** Let $a, b, c, d, e, f$ be integers (not necessarily distinct) between $-100$ and $100$, inclusive, such that $a + b + c + d + e + f = 100$. Let $M$ and $m$ be the maximum and minimum possible values, respectively, of
$$abc + bcd + cde + def + efa + fab + ace + bdf.$$
Find $\frac{M}{m}$.

**p15.** In quadrilateral $ABCD$, diagonal $AC$ bisects diagonal $BD$. Given that $AB = 20$, $BC = 15$, $CD = 13$, $AC = 25$, find $DA$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Guts Round

– Round 1

**p1.** Ravi has a bag with $100$ slips of paper in it. Each slip has one of the numbers $3, 5$, or $7$ written on it. Given that half of the slips have the number $3$ written on them, and the average of the values on all the slips is $4.4$, how many slips have $7$ written on them?

**p2.** In triangle $ABC$, point $D$ lies on side $AB$ such that $AB \perp CD$. It is given that $\frac{CD}{BD} = \frac{1}{2}$, $AC = 29$, and $AD = 20$. Find the area of triangle $BCD$.

**p3.** Compute $(123 + 4)(123 + 5) - 123 \cdot 132$.

Round 2

**p4.** David is evaluating the terms in the sequence $a_n = (n + 1)^3 - n^3$ for $n = 1, 2, 3, ....$ (that is, $a_1 = 2^3 - 1^3$ , $a_2 = 3^3 - 2^3$, $a_3 = 4^3 - 3^3$, and so on). Find the first composite number in the sequence. (An positive integer is composite if it has a divisor other than 1 and itself.)

**p5.** Find the sum of all positive integers strictly less than $100$ that are not divisible by $3$.

**p6.** In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).
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png

Round 3

**p7.** Fresh Mann is a $9$th grader at Euclid High School. Fresh Mann thinks that the word vertices is the plural of the word vertice. Indeed, vertices is the plural of the word vertex. Using all the letters in the word vertice, he can make $m$ $7$-letter sequences. Using all the letters in the word vertex, he can make $n$ $6$-letter sequences. Find $m - n$.

**p8.** Fresh Mann is given the following expression in his Algebra $1$ class: $101 - 102 = 1$. Fresh Mann is allowed to move some of the digits in this (incorrect) equation to make it into a correct equation. What is the minimal number of digits Fresh Mann needs to move?

**p9.** Fresh Mann said, "The function $f(x) = ax^2 + bx + c$ passes through $6$ points. Their $x$-coordinates are consecutive positive integers, and their y-coordinates are $34$, $55$, $84$, $119$, $160$, and $207$, respectively.” Sophy Moore replied, ”You’ve made an error in your list,” and replaced one of Fresh Mann’s numbers with the correct y-coordinate. Find the corrected value.

Round 4

**p10.** An assassin is trying to find his target’s hotel room number, which is a three-digit positive integer. He knows the following clues about the number:
(a) The sum of any two digits of the number is divisible by the remaining digit.
(b) The number is divisible by $3$, but if the first digit is removed, the remaining two-digit number is not.
(c) The middle digit is the only digit that is a perfect square.
Given these clues, what is a possible value for the room number?

**p11.** Find a positive real number $r$ that satisfies

$$\frac{4+r^3}{9+r^6} = \frac{1}{5-r^3} - \frac{1}{9+r^6}.$$

**p12.** Find the largest integer $n$ such that there exist integers $x$ and $y$ between $1$ and $20$ inclusive with

$$\left|\frac{21}{19} - \frac{x}{y}\right| < \frac{1}{n}.$$

PS. You had better use hide for answers. Last rounds have been posted here (`https://artofproblemsolvin` `com/community/c4h2784267p24464980`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

**p13.** A unit square is rotated $30^o$ counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

**p14.** Suppose points $A$ and $B$ lie on a circle of radius $4$ with center $O$, such that $\angle AOB = 90^o$. The perpendicular bisectors of segments $OA$ and $OB$ divide the interior of the circle into four regions. Find the area of the smallest region.

**p15.** Let $ABCD$ be a quadrilateral such that $AB = 4$, $BC = 6$, $CD = 5$, $DA = 3$, and $\angle DAB = 90^o$. There is a point $I$ inside the quadrilateral that is equidistant from all the sides. Find $AI$.

Round 6

 

*The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.*

**p16.** Let $C$ be the answer to problem $18$. Compute

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{C^2}\right).$$

**p17.** Let $A$ be the answer to problem $16$. Let $PQRS$ be a square, and let point $M$ lie on segment $PQ$ such that $MQ = 7PM$ and point $N$ lie on segment $PS$ such that $NS = 7PN$. Segments $MS$ and $NQ$ meet at point $X$. Given that the area of quadrilateral $PMXN$ is $A-\frac{1}{2}$, find the side length of the square.

**p18.** Let $B$ be the answer to problem $17$ and let $N = 6B$. Find the number of ordered triples $(a, b, c)$ of integers between $0$ and $N-1$, inclusive, such that $a+b+c$ is divisible by $N$.

Round 7

**p19.** Let $k$ be the units digit of $\underbrace{7^{7^{7^{7^{7^{7^{7}}}}}}}_{Seven\ 7s}$. What is the largest prime factor of the number consisting of $k$ 7's written in a row?

**p20.** Suppose that $E = 7^7$, $M = 7$, and $C = 777$. The characters $E, M, C, C$ are arranged randomly in the following blanks.

$$... \times ... \times ... \times ...$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

**p21**. During a recent math contest, Sophy Moore made the mistake of thinking that $133$ is a prime number. Fresh Mann replied, "To test whether a number is divisible by $3$, we just need to check whether the sum of the digits is divisible by $3$. By the same reasoning, to test whether a number is divisible by $7$, we just need to check that the sum of the digits is a multiple of $7$, so $133$ is clearly divisible by $7$." Although his general principle is false, $133$ is indeed divisible by $7$. How many three-digit numbers are divisible by $7$ and have the sum of their digits divisible by $7$?

Round 8

**p22.** A *look-and-say* sequence is defined as follows: starting from an initial term $a_1$, each subsequent term $a_k$ is found by reading the digits of $a_{k-1}$ from left to right and specifying the number of times each digit appears consecutively. For example, $4$ would be succeeded by $14$ ("One four."), and $31337$ would be followed by $13112317$ ("One three, one one, two three, one seven.") If $a_1$ is a random two-digit positive integer, find the probability that $a_4$ is at least six digits long.

**p23.** In triangle $ABC$, $\angle C = 90^o$. Point $P$ lies on segment $BC$ and is not $B$ or $C$. Point $I$ lies on segment $AP$, and $\angle BIP = \angle PBI = \angle CAB$. If $\frac{AP}{BC} = k$, express $\frac{IP}{CP}$ in terms of $k$.

**p24.** A subset of $\{1, 2, 3, ..., 30\}$ is called *delicious* if it does not contain an element that is $3$ times another element. A subset is called super delicious if it is delicious and no delicious set has more elements than it has. Determine the number of super delicious subsets.

PS. You sholud use hide for answers. First rounds have been posted here (https://artofproblemsolving.com/community/c4h2784267p24464980). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Accuracy

– **p1.** An $18$oz glass of apple juice is $6\%$ sugar and a $6$oz glass of orange juice is $12\%$ sugar. The two glasses are poured together to create a cocktail. What percent of the cocktail is sugar?

**p2.** Find the number of positive numbers that can be expressed as the difference of two integers between $-2$ and $2012$ inclusive.

**p3.** An annulus is defined as the region between two concentric circles. Suppose that the inner circle of an annulus has radius $2$ and the outer circle has radius $5$. Find the probability that a randomly chosen point in the annulus is at most $3$ units from the center.

**p4.** Ben and Jerry are walking together inside a train tunnel when they hear a train approaching. They decide to run in opposite directions, with Ben heading towards the train and Jerry heading away from the train. As soon as Ben finishes his $1200$ meter dash to the outside, the front of the train enters the tunnel. Coincidentally, Jerry also barely survives, with the front of the train exiting the tunnel as soon as he does. Given that Ben and Jerry both run at $1/9$ of the train's speed, how long is the tunnel in meters?

**p5.** Let $ABC$ be an isosceles triangle with $AB = AC = 9$ and $\angle B = \angle C = 75^o$. Let $DEF$ be another triangle congruent to $ABC$. The two triangles are placed together (without overlapping) to form a quadrilateral, which is cut along one of its diagonals into two triangles. Given that the two resulting triangles are incongruent, find the area of the larger one.

**p6.** There is an infinitely long row of boxes, with a Ditto in one of them. Every minute, each existing Ditto clones itself, and the clone moves to the box to the right of the original box, while the original Ditto does not move. Eventually, one of the boxes contains over $100$ Dittos. How many Dittos are in that box when this first happens?

**p7.** Evaluate
$$26 + 36 + 998 + 26 \cdot 36 + 26 \cdot 998 + 36 \cdot 998 + 26 \cdot 36 \cdot 998.$$

**p8.** There are $15$ students in a school. Every two students are either friends or not friends. Among every group of three students, either all three are friends with each other, or exactly one pair of them are friends. Determine the minimum possible number of friendships at the school.

**p9.** Let $f(x) = \sqrt{2x + 1 + 2\sqrt{x^2 + x}}$. Determine the value of
$$\frac{1}{f(1)} + \frac{1}{f(1)} + \frac{1}{f(3)} + ... + \frac{1}{f(24)}.$$

**p10.** In square $ABCD$, points $E$ and $F$ lie on segments $AD$ and $CD$, respectively. Given that $\angle EBF = 45^o$, $DE = 12$, and $DF = 35$, compute $AB$.

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– Individual Speed

– *20 problems for 20 minutes.*

**p1.** Evaluate $= \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5}$.

**p2.** A regular hexagon and a regular $n$-sided polygon have the same perimeter. If the ratio of the side length of the hexagon to the side length of the $n$-sided polygon is $2 : 1$, what is $n$?

**p3.** How many nonzero digits are there in the decimal representation of $2 \cdot 10 \cdot 500 \cdot 2500$?

**p4.** When the numerator of a certain fraction is increased by $2012$, the value of the fraction increases by $2$. What is the denominator of the fraction?

 

**p5.** Sam did the computation $1 - 10 \cdot a + 22$, where $a$ is some real number, except he messed up his order of operations and computed the multiplication last; that is, he found the value of $(1 - 10) \cdot (a + 22)$ instead. Luckily, he still ended up with the right answer. What is $a$?

**p6.** Let $n! = n \cdot (n-1) \cdots 2 \cdot 1$. For how many integers $n$ between $1$ and $100$ inclusive is $n!$ divisible by $36$?

**p7.** Simplify the expression $\sqrt{\frac{3 \cdot 27^3}{27 \cdot 3^3}}$

**p8.** Four points $A, B, C, D$ lie on a line in that order such that $\frac{AB}{CB} = \frac{AD}{CD}$ . Let $M$ be the midpoint of segment $AC$. If $AB = 6$, $BC = 2$, compute $MB \cdot MD$.

**p9.** Allan has a deck with $8$ cards, numbered $1$, $1$, $2$, $2$, $3$, $3$, $4$, $4$. He pulls out cards without replacement, until he pulls out an even numbered card, and then he stops. What is the probability that he pulls out exactly $2$ cards?

**p10.** Starting from the sequence $(3, 4, 5, 6, 7, 8, ...)$, one applies the following operation repeatedly. In each operation, we change the sequence

$$(a_1, a_2, a_3, ..., a_{a_1-1}, a_{a_1}, a_{a_1+1}, ...)$$

to the sequence

$$(a_2, a_3, ..., a_{a_1}, a_1, a_{a_1+1}, ...).$$

(In other words, for a sequence starting with$x$, we shift each of the next $x - 1$ term to the left by one, and put x immediately to the right of these numbers, and keep the rest of the terms unchanged. For example, after one operation, the sequence is $(4, 5, 3, 6, 7, 8, ...)$, and after two operations, the sequence becomes $(5, 3, 6, 4, 7, 8, ...)$. How many operations will it take to obtain a sequence of the form $(7, ...)$ (that is, a sequence starting with $7$)?

**p11.** How many ways are there to place $4$ balls into a $4 \times 6$ grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

**p12.** Point $P$ lies inside triangle $ABC$ such that $\angle PBC = 30^o$ and $\angle PAC = 20^o$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.

**p13.** What is the largest prime factor of $9^3 - 4^3$?

**p14.** Joey writes down the numbers $1$ through $10$ and crosses one number out. He then adds the remaining numbers. What is the probability that the sum is less than or equal to $47$?

**p15.** In the coordinate plane, a lattice point is a point whose coordinates are integers. There is a pile of grass at every lattice point in the coordinate plane. A certain cow can only eat piles of grass that are at most $3$ units away from the origin. How many piles of grass can she eat?

**p16.** A book has 1000 pages numbered $1$, $2$, ... , $1000$. The pages are numbered so that pages $1$ and $2$ are back to back on a single sheet, pages $3$ and $4$ are back to back on the next sheet, and so on, with pages $999$ and $1000$ being back to back on the last sheet. How many pairs of pages that are back to back (on a single sheet) share no digits in the same position? (For example, pages $9$ and $10$, and pages $89$ and $90$.)

**p17.** Find a pair of integers $(a, b)$ for which $\frac{10^a}{a!} = \frac{10^b}{b!}$ and $a < b$.

**p18.** Find all ordered pairs $(x, y)$ of real numbers satisfying

$$\begin{cases} -x^2 + 3y^2 - 5x + 7y + 4 = 0 \\ 2x^2 - 2y^2 - x + y + 21 = 0 \end{cases}$$

**p19.** There are six blank fish drawn in a line on a piece of paper. Lucy wants to color them either red or blue, but will not color two adjacent fish red. In how many ways can Lucy color the fish?

**p20.** There are sixteen $100$-gram balls and sixteen $99$-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the $32$ balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?

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