Exeter Math Club Competition 2013

www.artofproblemsolving.com/community/c3003272

by parmenides51

– Team Round

– **p1.** Determine the number of ways to place 4 rooks on a 4×4 chessboard such that:
(a) no two rooks attack one another, and
(b) the main diagonal (the set of squares marked X below) does not contain any rooks.
`https://cdn.artofproblemsolving.com/attachments/e/e/e3aa96de6c8ed468c6ef3837e66a0bce360d3`
`png`
The rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

p2. Seven students, numbered 1 to 7 in counter-clockwise order, are seated in a circle. Fresh Mann has 100 erasers, and he wants to distribute them to the students, albeit unfairly. Starting with person 1 and proceeding counter-clockwise, Fresh Mann gives i erasers to student i; for example, he gives 1 eraser to student 1, then 2 erasers to student 2, et cetera. He continues around the circle until he does not have enough erasers to give to the next person. At this point, determine the number of erasers that Fresh Mann has.

p3. Let ABC be a triangle with $AB = AC = 17$ and $BC = 24$. Approximate $\angle ABC$ to the nearest multiple of 10 degrees.

p4. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = \frac{3}{5}$ and for $n \ge 1$, $x_{n+1} = 2 - \frac{1}{x_n}$. Compute the product $x_1x_2x_3...x_{2013}$.

p5. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PR = 2$, $IM = 3$, and $ES = 5$, compute the area of hexagon $PRIMES$.

p6. Let $f(a, b) = \frac{a^2}{a+b}$. Let A denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \le i < j \le 10$; that is,

$$A = (f(1, 2) + f(1, 3) + ... + f(1, 10)) + (f(2, 3) + f(2, 4) + ... + f(2, 10)) + ... + f(9, 10).$$

Similarly, let B denote the sum of $f(i, j)$ over all pairs of integers (i, j) with $1 \le j < i \le 10$, that is,

$$B = (f(2, 1) + f(3, 1) + ... + f(10, 1)) + (f(3, 2) + f(4, 2) + ... + f(10, 2)) + ... + f(10, 9).$$

Compute $B - A$.

p7. Fresh Mann has a pile of seven rocks with weights $1, 1, 2, 4, 8, 16$, and 32 pounds and some integer X between 1 and 64, inclusive. He would like to choose a set of the rocks whose total weight is exactly X pounds. Given that he can do so in more than one way, determine the sum of all possible values of X. (The two 1-pound rocks are indistinguishable.)

p8. Let $ABCD$ be a convex quadrilateral with $AB = BC = CA$. Suppose that point P lies inside the quadrilateral with $AP = PD = DA$ and $\angle PCD = 30^o$. Given that $CP = 2$ and $CD = 3$, compute CA.

p9. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = 2$, $x_2 = \frac{13}{2}$, and for $n \ge 1$, $x_{n+2} = 3 - \frac{3}{x_{n+1}} + \frac{1}{x_nx_{n+1}}$. Compute x_{100}.

p10. Ten prisoners are standing in a line. A prison guard wants to place a hat on each prisoner. He has two colors of hats, red and blue, and he has 10 hats of each color. Determine the number of ways in which the prison guard can place hats such that among any set of consecutive prisoners, the number of prisoners with red hats and the number of prisoners with blue hats differ by at most 2.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Guts Round

– Round 1

p1. Five girls and three boys are sitting in a room. Suppose that four of the children live in California. Determine the maximum possible number of girls that could live somewhere outside California.

p2. A 4-meter long stick is rotated 60^o about a point on the stick 1 meter away from one of its ends. Compute the positive difference between the distances traveled by the two endpoints of the stick, in meters.

p3. Let $f(x) = 2x(x - 1)^2 + x^3(x - 2)^2 + 10(x - 1)^3(x - 2)$. Compute $f(0) + f(1) + f(2)$.

Round 2

p4. Twenty boxes with weights $10, 20, 30, ..., 200$ pounds are given. One hand is needed to lift a box for every 10 pounds it weighs. For example, a 40 pound box needs four hands to be lifted. Determine the number of people needed to lift all the boxes simultaneously, given that no person can help lift more than one box at a time.

p5. Let ABC be a right triangle with a right angle at A, and let D be the foot of the perpendicular from vertex$A$ to side BC. If $AB = 5$ and $BC = 7$, compute the length of segment AD.

p6. There are two circular ant holes in the coordinate plane. One has center $(0, 0)$ and radius 3, and the other has center $(20, 21)$ and radius 5. Albert wants to cover both of them completely with a circular bowl. Determine the minimum possible radius of the circular bowl.

Round 3

p7. A line of slope -4 forms a right triangle with the positive x and y axes. If the area of the triangle is 2013, find the square of the length of the hypotenuse of the triangle.

p8. Let ABC be a right triangle with a right angle at B, $AB = 9$, and $BC = 7$. Suppose that point P lies on segment AB with $AP = 3$ and that point Q lies on ray BC with $BQ = 11$. Let segments AC and PQ intersect at point X. Compute the positive difference between the areas of triangles APX and CQX.

p9. Fresh Mann and Sophy Moore are racing each other in a river. Fresh Mann swims downstream, while Sophy Moore swims $\frac{1}{2}$ mile upstream and then travels downstream in a boat. They start at the same time, and they reach the finish line 1 mile downstream of the starting point simultaneously. If Fresh Mann and Sophy Moore both swim at 1 mile per hour in still water and the boat travels at 10 miles per hour in still water, find the speed of the current.

Round 4

p10. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and for $n \ge 1$, $F_{n+1} = F_n + F_{n-1}$. The first few terms of the Fibonacci sequence are 0, 1, 1, 2, 3, 5, 8, 13. Every positive integer can be expressed as the sum of nonconsecutive, distinct, positive Fibonacci numbers, for example, $7 = 5 + 2$. Express 121 as the sum of nonconsecutive, distinct, positive Fibonacci numbers. (It is not permitted to use both a 2 and a 1 in the expression.)

p11. There is a rectangular box of surface area 44 whose space diagonals have length 10. Find the sum of the lengths of all the edges of the box.

p12. Let ABC be an acute triangle, and let D and E be the feet of the altitudes to BC and CA, respectively. Suppose that segments AD and BE intersect at point H with $AH = 20$ and $HD = 13$. Compute $BD \cdot CD$.

PS. You should use hide for answers. Rounds 5-8 have been posted here (`https://artofproblemsolving.com/community/c4h2809420p24782524`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. In coordinate space, a lattice point is a point all of whose coordinates are integers. The lattice points (x, y, z) in three-dimensional space satisfying $0 \le x, y, z \le 5$ are colored in n colors such that any two points that are $\sqrt{3}$ units apart have different colors. Determine the minimum possible value of n.

p14. Determine the number of ways to express 121 as a sum of strictly increasing positive Fibonacci numbers.

p15. Let $ABCD$ be a rectangle with $AB = 7$ and $BC = 15$. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed outside the rectangle. Compute the area of quadrilateral $PQRS$.

Round 6

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

p16. Let C be the answer to problem 18. Suppose that x and y are real numbers with $y > 0$ and

$$x + y = C$$

$$x + \frac{1}{y} = -2.$$

Compute $y + \frac{1}{y}$.

p17. Let A be the answer to problem 16. Let PQR be a triangle with $\angle PQR = 90^o$, and let X be the foot of the perpendicular from point Q to segment PR. Given that $QX = A$, determine the minimum possible area of triangle PQR.

p18. Let B be the answer to problem 17 and let $K = 36B$. Alice, Betty, and Charlize are identical triplets, only distinguishable by their hats. Every day, two of them decide to exchange hats. Given that they each have their own hat today, compute the probability that Alice will have her own hat in K days.

Round 7

p19. Find the number of positive integers a such that all roots of $x^2 + ax + 100$ are real and the sum of their squares is at most 2013.

p20. Determine all values of k such that the system of equations

$$y = x^2 - kx + 1$$

$$x = y^2 - ky + 1$$

has a real solution.

p21. Determine the minimum number of cuts needed to divide an $11 \times 5 \times 3$ block of chocolate into $1 \times 1 \times 1$ pieces. (When a block is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

Round 8

p22. A sequence that contains the numbers $1, 2, 3, ..., n$ exactly once each is said to be a permutation of length n. A permutation $w_1w_2w_3...w_n$ is said to be sad if there are indices $i < j < k$ such that $w_j > w_k$ and $w_j > w_i$. For example, the permutation 3142756 is sad because $7 > 6$ and $7 > 1$. Compute the number of permutations of length 11 that are not sad.

p23. Let ABC be a triangle with $AB = 39$, $BC = 56$, and $CA = 35$. Compute $\angle CAB - \angle ABC$ in degrees.

p24. On a strange planet, there are n cities. Between any pair of cities, there can either be a one-way road, two one-way roads in different directions, or no road at all. Every city has a name, and at the source of every one-way road, there is a signpost with the name of the destination city. In addition, the one-way roads only intersect at cities, but there can be bridges to prevent intersections at non-cities. Fresh Mann has been abducted by one of the aliens, but Sophy Moore knows that he is in Rome, a city that has no roads leading out of it. Also, there is a direct one-way road leading from each other city to Rome. However, Rome is the secret police's name for the so-described city; its official name, the name appearing on the labels of the one-way roads, is unknown to Sophy Moore. Sophy Moore is currently in Athens and she wants to head to Rome in order to rescue Fresh Mann, but she does not know the value of n. Assuming that she tries to minimize the number of roads on which she needs to travel, determine the maximum possible number of roads that she could be forced to travel in order to find Rome. Express your answer as a function of n.

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c4h2809419p24782489). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Accuracy

– **p1.** Find the largest possible number of consecutive 9's in which an integer between $10,000,000$ and $13,371,337$ can end. For example, 199 ends in two 9's, while $92,999$ ends in three 9's.

p2. Let $ABCD$ be a square of side length 2. Equilateral triangles ABP, BCQ, CDR, and DAS are constructed inside the square. Compute the area of quadrilateral $PQRS$.

p3. Evaluate the expression $7 \cdot 11 \cdot 13 \cdot 1003 - 3 \cdot 17 \cdot 59 \cdot 331$.

p4. Compute the number of positive integers c such that there is a non-degenerate obtuse triangle with side lengths 21, 29, and c.

p5. Consider a 5×5 board, colored like a chessboard, such that the four corners are black. Determine the number of ways to place 5 rooks on black squares such that no two of the rooks attack one another, given that the rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

p6. Let $ABCD$ be a trapezoid of height 6 with bases AB and CD. Suppose that $AB = 2$ and $CD = 3$, and let F and G be the midpoints of segments AD and BC, respectively. If diagonals AC and BD intersect at point E, compute the area of triangle FGE.

p7. A regular octahedron is a solid with eight faces that are congruent equilateral triangles. Suppose that an ant is at the center of one face of a regular octahedron of edge length 10. The ant wants to walk along the surface of the octahedron to reach the center of the opposite face. (Two faces of an octahedron are said to be opposite if they do not share a vertex.) Determine the minimum possible distance that the ant must walk.

p8. Let $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, and $D_1D_2D_3$ be triangles in the plane. All the sides of the four triangles are extended into lines. Determine the maximum number of pairs of these lines that can meet at 60^o angles.

p9. For an integer n, let $f_n(x)$ denote the function $f_n(x) = \sqrt{x^2 - 2012x + n} + 1006$. Determine all positive integers a such that $f_a(f_{2012}(x)) = x$ for all $x \geq 2012$.

p10. Determine the number of ordered triples of integers (a, b, c) such that $(a+b)(b+c)(c+a) = 1800$.

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– Individual Speed

– *20 problems for 20 minutes.*

p1. Determine how many digits the number 10^{10} has.

p2. Let ABC be a triangle with $\angle ABC = 60^o$ and $\angle BCA = 70^o$. Compute $\angle CAB$ in degrees.

p3. Given that $x : y = 2012 : 2$ and $y : z = 1 : 2013$, compute $x : z$. Express your answer as a common fraction.

p4. Determine the smallest perfect square greater than 2400.

p5. At $12 : 34$ and $12 : 43$, the time contains four consecutive digits. Find the next time after 12:43 that the time contains four consecutive digits on a 24-hour digital clock.

p6. Given that $\sqrt{3^a \cdot 9^a \cdot 3^a} = 81^2$, compute a.

p7. Find the number of positive integers less than 8888 that have a tens digit of 4 and a units digit of 2.

p8. Find the sum of the distinct prime divisors of $1 + 2012 + 2013 + 2011 \cdot 2013$.

p9. Albert wants to make 2×3 wallet sized prints for his grandmother. Find the maximum possible number of prints Albert can make using one 4×7 sheet of paper.

p10. Let ABC be an equilateral triangle, and let D be a point inside ABC. Let E be a point such that ADE is an equilateral triangle and suppose that segments DE and AB intersect at point F. Given that $\angle CAD = 15^o$, compute $\angle DFB$ in degrees.

p11. A palindrome is a number that reads the same forwards and backwards; for example, 1221 is a palindrome. An almost-palindrome is a number that is not a palindrome but whose first and last digits are equal; for example, 1231 and 1311 are an almost-palindromes, but 1221 is not. Compute the number of 4-digit almost-palindromes.

p12. Determine the smallest positive integer n such that the sum of the digits of 11^n is not 2^n.

p13. Determine the minimum number of breaks needed to divide an 8×4 bar of chocolate into 1×1pieces. (When a bar is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

p14. A particle starts moving on the number line at a time $t = 0$. Its position on the number line, as a function of time, is

$$x = (t - 2012)^2 - 2012(t - 2012) - 2013.$$

Find the number of positive integer values of t at which time the particle lies in the negative half of the number line (strictly to the left of 0).

p15. Let A be a vertex of a unit cube and let B,C, and D be the vertices adjacent to A. The tetrahedron $ABCD$ is cut off the cube. Determine the surface area of the remaining solid.

p16. In equilateral triangle ABC, points P and R lie on segment AB, points I and M lie on segment BC, and points E and S lie on segment CA such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PS = 2$, $RI = 3$, and $ME = 5$, compute the area of hexagon $PRIMES$.

p17. Find the smallest odd positive integer with an odd number of positive integer factors, an odd number of distinct prime factors, and an odd number of perfect square factors.

p18. Fresh Mann thinks that the expressions $2\sqrt{x^2 - 4}$and $2(\sqrt{x^2} - \sqrt{4})$ are equivalent to each other, but the two expressions are not equal to each other for most real numbers x. Find all real numbers x such that $2\sqrt{x^2-4} = 2(\sqrt{x^2}-\sqrt{4})$.

p19. Let m be the positive integer such that a 3×3 chessboard can be tiled by at most m pairwise incongruent rectangles with integer side lengths. If rotations and reflections of tilings are considered distinct, suppose that there are n ways to tile the chessboard with m pairwise incongruent rectangles with integer side lengths. Find the product mn.

p20. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. A triangle XYZ is said to be friendly if it intersects triangle ABC and it is a translation of triangle ABC. Let S be the set of points in the plane that are inside some friendly triangle. Compute the ratio of the area of S to the area of triangle ABC.

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