Exeter Math Club Competition 2017
www.artofproblemsolving.com/community/c3003280
by parmenides51

– Team Round

– **p1.** Compute $2017 + 7201 + 1720 + 172$.

p2. A number is called *downhill* if its digits are distinct and in descending order. (For example, 653 and 8762 are downhill numbers, but 97721 is not.) What is the smallest downhill number greater than 86432?

p3. Each vertex of a unit cube is sliced off by a planar cut passing through the midpoints of the three edges containing that vertex. What is the ratio of the number of edges to the number of faces of the resulting solid?

p4. In a square with side length 5, the four points that divide each side into five equal segments are marked. Including the vertices, there are 20 marked points in total on the boundary of the square. A pair of distinct points A and B are chosen randomly among the 20 points. Compute the probability that $AB = 5$.

p5. A positive two-digit integer is one less than five times the sum of its digits. Find the sum of all possible such integers.

p6. Let
$$f(x) = 5^{4^{3^{2^x}}}.$$
Determine the greatest possible value of L such that $f(x) > L$ for all real numbers x.

p7. If $\overline{AAAA} + \overline{BB} = \overline{ABCD}$ for some distinct base-10 digits A, B, C, D that are consecutive in some order, determine the value of $ABCD$. (The notation $\overline{ABCD}$ refers to the four-digit integer with thousands digit A, hundreds digit B, tens digit C, and units digit D.)

p8. A regular tetrahedron and a cube share an inscribed sphere. What is the ratio of the volume of the tetrahedron to the volume of the cube?

p9. Define $\lfloor x \rfloor$ as the greatest integer less than or equal to x, and $x = x - \lfloor x \rfloor$ as the fractional part of x. If $\lfloor x^2 \rfloor = 2\lfloor x \rfloor$ and $\{x^2\} = \frac{1}{2}\{x\}$, determine all possible values of x.

p10. Find the largest integer $N > 1$ such that it is impossible to divide an equilateral triangle of side length 1 into N smaller equilateral triangles (of possibly different sizes).

p11. Let f and g be two quadratic polynomials. Suppose that f has zeroes 2 and 7, g has zeroes 1 and 8, and $f - g$ has zeroes 4 and 5. What is the product of the zeroes of the polynomial $f + g$?

p12. In square $PQRS$, points A, B, C, D, E, and F are chosen on segments PQ, QR, PR, RS, SP, and PR, respectively, such that $ABCDEF$ is a regular hexagon. Find the ratio of the area of $ABCDEF$ to the area of $PQRS$.

p13. For positive integers m and n, define $f(m, n)$ to be the number of ways to distribute m identical candies to n distinct children so that the number of candies that any two children receive differ by at most 1. Find the number of positive integers n satisfying the equation $f(2017, n) = f(7102, n)$.

p14. Suppose that real numbers x and y satisfy the equation

$$x^4 + 2x^2y^2 + y^4 - 2x^2 + 32xy - 2y^2 + 49 = 0.$$

Find the maximum possible value of $\frac{y}{x}$.

p15. A point P lies inside equilateral triangle ABC. Let A', B', C' be the feet of the perpendiculars from P to BC, AC, AB, respectively. Suppose that $PA = 13$, $PB = 14$, and $PC = 15$. Find the area of $A'B'C'$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Guts Round

– Round 1

p1. If $2m = 200cm$ and $m \neq 0$, find c.

p2. A right triangle has two sides of lengths 3 and 4. Find the smallest possible length of the third side.

p3. Given that $20(x + 17) = 17(x + 20)$, determine the value of x.

Round 2

p4. According to the Egyptian Metropolitan Culinary Community, food service is delayed on $\frac{2}{3}$ of flights departing from Cairo airport. On average, if flights with delayed food service have twice as many passengers per flight as those without, what is the probability that a passenger departing from Cairo airport experiences delayed food service?

p5. In a positive geometric sequence $\{a_n\}$, $a_1 = 9$, $a_9 = 25$. Find the integer k such that $a_k = 15$

p6. In the Delicate, Elegant, and Exotic Music Organization, pianist Hans is selling two types of owers with different prices (per ower): magnolias and myosotis. His friend Alice originally plans to buy a bunch containing 50% more magnolias than myosotis for $\$50$, but then she realizes that if she buys 50% less magnolias and 50% more myosotis than her original plan, she would still need to pay the same amount of money. If instead she buys 50% more magnolias and 50% less myosotis than her original plan, then how much, in dollars, would she need to pay?

Round 3

p7. In square $ABCD$, point P lies on side AB such that $AP = 3$,$BP = 7$. Points Q, R, S lie on sides BC, CD, DA respectively such that $PQ = PR = PS = AB$. Find the area of quadrilateral $PQRS$.

p8. Kristy is thinking of a number $n < 10^4$ and she says that 143 is one of its divisors. What is the smallest number greater than 143 that could divide n?

p9. A positive integer n is called *special* if the product of the n smallest prime numbers is divisible by the sum of the n smallest prime numbers. Find the sum of the three smallest special numbers.

Round 4

p10. In the diagram below, all adjacent points connected with a segment are unit distance apart. Find the number of squares whose vertices are among the points in the diagram and whose sides coincide with the drawn segments.
`https://cdn.artofproblemsolving.com/attachments/b/a/923e4d2d44e436ccec90661648967908306fe`
`png`

p11. Geyang tells Junze that he is thinking of a positive integer. Geyang gives Junze the following clues: • My number has three distinct odd digits. • It is divisible by each of its three digits, as well as their sum.
What is the sum of all possible values of Geyang's number?

p12. Regular octagon $ABCDEFGH$ has center O and side length 2. A circle passes through A, B, and O. What is the area of the part of the circle that lies outside of the octagon?

PS. You should use hide for answers. Rounds 5-8 have been posted here (`https://artofproblemsolving.com/community/c3h2936505p26278645`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– *Round 5*

p13. Kelvin Amphibian, a not-frog who lives on the coordinate plane, likes jumping around. Each step, he jumps either to the spot that is 1 unit to the right and 2 units up, or the spot that is 2 units to the right and 1 unit up, from his current location. He chooses randomly among these two choices with equal probability. He starts at the origin and jumps for a long time. What is the probability that he lands on $(10, 8)$ at some time in his journey?

p14. Points A, B, C, and D are randomly chosen on the circumference of a unit circle. What is the probability that line segments AB and CD intersect inside the circle?

p15. Let $P(x)$ be a quadratic polynomial with two consecutive integer roots. If it is also known that $\frac{P(2017)}{P(2016)} = \frac{2016}{2017}$, find the larger root of $P(x)$.

Round 6

p16. Let S_n be the sum of reciprocals of the integers between 1 and n inclusive. Find a triple (a, b, c) of positive integers such that $S_{2017} \cdot S_{2017} - S_{2016} \cdot S_{2018} = \frac{S_a+S_b}{c}$.

p17. Suppose that m and n are both positive integers. Alec has m standard 6-sided dice, each labelled 1 to 6 inclusive on the sides, while James has n standard 12-sided dice, each labelled 1 to 12 inclusive on the sides. They decide to play a game with their dice. They each toss all their dice simultaneously and then compute the sum of the numbers that come up on their dice. Whoever has a higher sum wins (if the sums are equal, they tie). Given that both players have an equal chance of winning, determine the minimum possible value of mn.

p18. Overlapping rectangles $ABCD$ and $BEDF$ are congruent to each other and both have area 1. Given that A, C, E, F are the vertices of a square, find the area of the square.

Round 7

p19. Find the number of solutions to the equation

$$|||...|||||x| + 1| - 2| + 3| - 4| + ... - 98| + 99| - 100| = 0$$

p20. A split of a positive integer in base 10 is the separation of the integer into two nonnegative integers, allowing leading zeroes. For example, 2017 can be split into 2 and 017 (or 17), 20 and 17, or 201 and 7. A split is called squarish if both integers are nonzero perfect squares. 49 and 169 are the two smallest perfect squares that have a squarish split (4 and 9, 16 and 9 respectively). Determine all other perfect squares less than 2017 with at least one squarish split.

p21. Polynomial $f(x) = 2x^3 + 7x^2 - 3x + 5$ has zeroes a, b and c. Cubic polynomial $g(x)$ with x^3-coefficient 1 has zeroes a^2, b^2 and $c2$. Find the sum of coefficients of $g(x)$.

Round 8

p22. Two congruent circles, ω_1 and ω_2, intersect at points A and B. The centers of ω_1 and ω_2 are O_1 and O_2 respectively. The arc AB of ω_1 that lies inside ω_2 is trisected by points P and Q, with the points lying in the order A, P, Q, B. Similarly, the arc AB of ω_2 that lies inside ω_1 is trisected by points R and S, with the points lying in the order A, R, S, B. Given that $PQ = 1$ and $PR = \sqrt{2}$, find the measure of $\angle AO_1B$ in degrees.

p23. How many ordered triples of (a, b, c) of integers between -10 and 10 inclusive satisfy the equation $-abc = (a + b)(b + c)(c + a)$?

p24. For positive integers n and b where $b > 1$, define $s_b(n)$ as the sum of digits in the base-b representation of n. A positive integer p is said to dominate another positive integer q if for all positive integers n, $s_p(n)$ is greater than or equal to $s_q(n)$. Find the number of ordered pairs (p, q) of distinct positive integers between 2 and 100 inclusive such that p dominates q.

PS. You should use hide for answers. Rounds 1-5 have been posted here (https://artofproblemsolving.com/community/c3h2936487p26278546). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Accuracy

– **p1.** Chris goes to Matt's Hamburger Shop to buy a hamburger. Each hamburger must contain exactly one bread, one lettuce, one cheese, one protein, and at least one condiment. There are two kinds of bread, two kinds of lettuce, three kinds of cheese, three kinds of protein, and six different condiments: ketchup, mayo, mustard, dill pickles, jalape nos, and Matt's Magical Sunshine Sauce. How many different hamburgers can Chris make?

p2. The degree measures of the interior angles in convex pentagon $NICKY$ are all integers and form an increasing arithmetic sequence in some order. What is the smallest possible degree measure of the pentagon's smallest angle?

p3. Daniel thinks of a two-digit positive integer x. He swaps its two digits and gets a number y that is less than x. If 5 divides $x-y$ and 7 divides $x+y$, find all possible two-digit numbers Daniel could have in mind.

p4. At the Lio Orympics, a target in archery consists of ten concentric circles. The radii of the circles are 1, 2, 3, ..., 9, and 10 respectively. Hitting the innermost circle scores the archer 10 points, the next ring is worth 9 points, the next ring is worth 8 points, all the way to the outermost ring, which is worth 1 point. If a beginner archer has an equal probability of hitting any point on the target and never misses the target, what is the probability that his total score after making two shots is even?

p5. Let $F(x)=x^2+2x-35$ and $G(x)=x^2+10x+14$. Find all distinct real roots of $F(G(x))=0$.

p6. One day while driving, Ivan noticed a curious property on his car's digital clock. The sum of the digits of the current hour equaled the sum of the digits of the current minute. (Ivan's car clock shows 24-hour time; that is, the hour ranges from 0 to 23, and the minute ranges from 0 to 59.) For how many possible times of the day could Ivan have observed this property?

p7. Qi Qi has a set Q of all lattice points in the coordinate plane whose x- and y-coordinates are between 1 and 7 inclusive. She wishes to color 7 points of the set blue and the rest white so that each row or column contains exactly 1 blue point and no blue point lies on or below the line $x+y=5$. In how many ways can she color the points?

p8. A piece of paper is in the shape of an equilateral triangle ABC with side length 12. Points A_B and B_A lie on segment AB, such that $AA_B=3$, and $BB_A=3$. Define points B_C and C_B on segment BC and points C_A and A_C on segment CA similarly. Point A_1 is the intersection of A_CB_C and A_BC_B. Define B_1 and C_1 similarly. The three rhombi - $AA_BA_1A_C$,$BB_CB_1B_A$, $CC_AC_1C_B$ -

are cut from triangle ABC, and the paper is folded along segments A_1B_1, B_1C_1, C_1A_1, to form a tray without a top. What is the volume of this tray?

p9. Define $\{x\}$ as the fractional part of x. Let S be the set of points (x, y) in the Cartesian coordinate plane such that $x + \{x\} \le y$, $x \ge 0$, and $y \le 100$. Find the area of S.

p10. Nicky likes dolls. He has 10 toy chairs in a row, and he wants to put some indistinguishable dolls on some of these chairs. (A chair can hold only one doll.) He doesn't want his dolls to get lonely, so he wants each doll sitting on a chair to be adjacent to at least one other doll. How many ways are there for him to put any number (possibly none) of dolls on the chairs? Two ways are considered distinct if and only if there is a chair that has a doll in one way but does not have one in the other.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Speed

– *20 problems for 25 minutes.*

p1. Ben was trying to solve for x in the equation $6 + x = 1$. Unfortunately, he was reading upside-down and misread the equation as $1 = x + 9$. What is the positive difference between Ben's answer and the correct answer?

p2. Anjali and Meili each have a chocolate bar shaped like a rectangular box. Meili's bar is four times as long as Anjali's, while Anjali's is three times as wide and twice as thick as Meili's. What is the ratio of the volume of Anjali's chocolate to the volume of Meili's chocolate?

p3. For any two nonnegative integers m, n, not both zero, define $m?n = m^n + n^m$. Compute the value of $((2?0)?1)?7$.

p4. Eliza is making an in-scale model of the Phillips Exeter Academy library, and her prototype is a cube with side length 6 inches. The real library is shaped like a cube with side length 120 feet, and it contains an entrance chamber in the front. If the chamber in Eliza's model is 0.8 inches wide, how wide is the real chamber, in feet?

p5. One day, Isaac begins sailing from Marseille to New York City. On the exact same day, Evan begins sailing from New York City to Marseille along the exact same route as Isaac. If Marseille and New York are exactly 3000 miles apart, and Evan sails exactly 40 miles per day, how many miles must Isaac sail each day to meet Evan's ship in 30 days?

p6. The conversion from Celsius temperature C to Fahrenheit temperature F is:

$$F = 1.8C + 32.$$

If the lowest temperature at Exeter one day was 20^o F, and the next day the lowest temperature was 5^o C higher, what would be the lowest temperature that day, in degrees Fahrenheit?

p7. In a school, 60% of the students are boys and 40% are girls. Given that 40% of the boys like math and 50% of the people who like math are girls, what percentage of girls like math?

p8. Adam and Victor go to an ice cream shop. There are four sizes available (kiddie, small, medium, large) and seventeen different flavors, including three that contain chocolate. If Victor insists on getting a size at least as large as Adam's, and Adam refuses to eat anything with chocolate, how many different ways are there for the two of them to order ice cream?

p9. There are 10 (not necessarily distinct) positive integers with arithmetic mean 10. Determine the maximum possible range of the integers. (The range is defined to be the nonnegative difference between the largest and smallest number within a list of numbers.)

p10. Find the sum of all distinct prime factors of $11! - 10! + 9!$.

p11. Inside regular hexagon $ZUMING$, construct square $FENG$. What fraction of the area of the hexagon is occupied by rectangle $FUME$?

p12. How many ordered pairs (x, y) of nonnegative integers satisfy the equation $4^x \cdot 8^y = 16^{10}$?

p13. In triangle ABC with $BC = 5$, $CA = 13$, and $AB = 12$, Points E and F are chosen on sides AC and AB, respectively, such that $EF \parallel BC$. Given that triangle AEF and trapezoid $EFBC$ have the same perimeter, find the length of EF.

p14. Find the number of two-digit positive integers with exactly 6 positive divisors. (Note that 1 and n are both counted among the divisors of a number n.)

p15. How many ways are there to put two identical red marbles, two identical green marbles, and two identical blue marbles in a row such that no red marble is next to a green marble?

p16. Every day, Yannick submits 8 more problems to the EMCC problem database than he did the previous day. Every day, Vinjai submits twice as many problems to the EMCC problem database as he did the previous day. If Yannick and Vinjai initially both submit one problem to the database on a Monday, on what day of the week will the total number of Vinjai's problems first exceed the total number of Yannick's problems?

p17. The tiny island nation of Konistan is a cone with height twelve meters and base radius nine meters, with the base of the cone at sea level. If the sea level rises four meters, what is the surface area of Konistan that is still above water, in square meters?

p18. Nicky likes to doodle. On a convex octagon, he starts from a random vertex and doodles a path, which consists of seven line segments between vertices. At each step, he chooses a vertex randomly among all unvisited vertices to visit, such that the path goes through all eight vertices and does not visit the same vertex twice. What is the probability that this path does not cross itself?

p19. In a right-angled trapezoid $ABCD$, $\angle B = \angle C = 90^o$, $AB = 20$, $CD = 17$, and $BC = 37$. A line perpendicular to DA intersects segment BC and DA at P and Q respectively and separates the trapezoid into two quadrilaterals with equal area. Determine the length of BP.

p20. A sequence of integers a_i is defined by $a_1 = 1$ and $a_{i+1} = 3i - 2a_i$ for all integers $i \ge 1$. Given that $a_{15} = 5476$, compute the sum $a_1 + a_2 + a_3 + ... + a_{15}$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).