Exeter Math Club Competition 2020

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by parmenides51

– Team Round

– **p1.** The number 2020 is very special: the sum of its digits is equal to the product of its nonzero digits. How many such four digit numbers are there? (Numbers with only one nonzero digit, like 3000, also count)

p2. A locker has a combination which is a sequence of three integers between 0 and 49, inclusive. It is known that all of the numbers in the combination are even. Let the total of a lock combination be the sum of the three numbers. Given that the product of the numbers in the combination is 12160, what is the sum of all possible totals of the locker combination?

p3. Given points $A = (0, 0)$ and $B = (0, 1)$ in the plane, the set of all points P in the plane such that triangle ABP is isosceles partitions the plane into k regions. The sum of the areas of those regions that are bounded is s. Find ks.

p4. Three families sit down around a circular table, each person choosing their seat at random. One family has two members, while the other two families have three members. What is the probability that every person sits next to at least one person from a different family?

p5. Jacob and Alexander are walking up an escalator in the airport. Jacob walks twice as fast as Alexander, who takes 18 steps to arrive at the top. Jacob, however, takes 27 steps to arrive at the top. How many of the upward moving escalator steps are visible at any point in time?

p6. Points A, B, C, D, E lie in that order on a circle such that $AB = BC = 5$, $CD = DE = 8$, and $\angle BCD = 150^o$. Let AD and BE intersect at P. Find the area of quadrilateral $PBCD$.

p7. Ivan has a triangle of integers with one number in the first row, two numbers in the second row, and continues up to eight numbers in the eighth row. He starts with the first 8 primes, 2 through 19, in the bottom row. Each subsequent row is filled in by writing the least common multiple of two adjacent numbers in the row directly below. For example, the second last row starts with$6, 15, 35$, etc. Let P be the product of all the numbers in this triangle. Suppose that P is a multiple of a/b, where a and b are positive integers and $a > 1$. Given that b is maximized, and for this value of b, a is also maximized, find $a + b$.

p8. Let $ABCD$ be a cyclic quadrilateral. Given that triangle ABD is equilateral, $\angle CBD = 15^o$, and $AC = 1$, what is the area of $ABCD$?

p9. Let S be the set of all integers greater than 1. The function f is defined on S and each value of f is in S. Given that f is nondecreasing and $f(f(x)) = 2x$ for all x in S, find $f(100)$.

p10. An *origin-symmetric* parallelogram P (that is, if (x, y) is in P, then so is $(-x, -y)$) lies in the coordinate plane. It is given that P has two horizontal sides, with a distance of 2020 between them, and that there is no point with integer coordinates except the origin inside P. Also, P has the maximum possible area satisfying the above conditions. The coordinates of the four vertices of P are $(a, 1010)$, $(b, 1010)$, $(-a, -1010)$, $(-b, -1010)$, where a, b are positive real numbers with $a < b$. What is b?

p11. What is the remainder when $5^{200} + 5^{50} + 2$ is divided by $(5 + 1)(5^2 + 1)(5^4 + 1)$?

p12. Let $f(n) = n^2 - 4096n - 2045$. What is the remainder when $f(f(f(...f(2046)...)))$ is divided by 2047, where the function f is applied 47 times?

p13. What is the largest possible area of a triangle that lies completely within a 97-dimensional hypercube of side length 1, where its vertices are three of the vertices of the hypercube?

p14. Let $N = \left\lfloor \frac{1}{61} \right\rfloor + \left\lfloor \frac{3}{61} \right\rfloor + \left\lfloor \frac{3^2}{61} \right\rfloor + ... + \left\lfloor \frac{3^{2019}}{61} \right\rfloor$. Given that $122N$ can be expressed as $3^a - b$, where a, b are positive integers and a is as large as possible, find $a + b$.

Note: $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x.

p15. Among all ordered triples of integers (x, y, z) that satisfy $x+y+z = 8$ and $x^3+y^3+z^3 = 134$, what is the maximum possible value of $|x| + |y| + |z|$?

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Accuracy

– **p1.** What is $(2 + 4 + ... + 20) - (1 + 3 + ... + 19)$?

p2. Two ants start on opposite vertices of a dodecagon (12-gon). Each second, they randomly move to an adjacent vertex. What is the probability they meet after four moves?

p3. How many distinct 8-letter strings can be made using 8 of the 9 letters from the words $FORK$ and $KNIFE$ (e.g., $FORKNIFE$)?

p4. Let ABC be an equilateral triangle with side length 8 and let D be a point on segment BC such that $BD = 2$. Given that E is the midpoint of AD, what is the value of $CE^2 - BE^2$?

(mistyped p4) Let ABC be an equilateral triangle with side length 8 and let D be a point on segment BC such that $BD = 2$. Given that E is the midpoint of AD, what is the value of $CE^2 + BE^2$?

p5. You have two fair six-sided dice, one labeled 1 to 6, and for the other one, each face is labeled 1, 2, 3, or 4 (not necessarily all numbers are used). Let p be the probability that when the two dice are rolled, the number on the special die is smaller than the number on the normal die. Given that $p = 1/2$, how many distinct combinations of 1, 2, 3, 4 can appear on the special die? The arrangement of the numbers on the die does not matter.

p6. Let ω_1 and ω_2 be two circles with centers A and B and radii 3 and 13, respectively. Suppose $AB = 10$ and that C is the midpoint of AB. Let ℓ be a line that passes through C and is tangent to ω_1 at P. Given that ℓ intersects ω_2 at X and Y such that $XP < YP$, what is XP?

p7. Let $f(x)$ be a cubic polynomial. Given that $f(1) = 13$, $f(4) = 19$, $f(7) = 7$, and $f(10) = 13$, find $f(13)$.

p8. For all integers $0 \le n \le 202$ not divisible by seven, define $f(n) = \{\sqrt{7n}\}$. For what value n does $f(n)$ take its minimum value? (Note: $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

p9. Let ABC be a triangle with $AB = 14$ and $AC = 25$. Let the incenter of ABC be I. Let line AI intersect the circumcircle of BIC at D (different from I). Given that line DC is tangent to the circumcircle of ABC, find the area of triangle BCD.

p10. Evaluate the infinite sum

$$\frac{4^2+3}{1\cdot 3\cdot 5\cdot 7} + \frac{6^2+3}{3\cdot 5\cdot 7\cdot 9} + \frac{8^2+3}{5\cdot 7\cdot 9\cdot 11} + ...$$

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– Individual Speed

– *20 problems for 25 minutes.*

p1. What is $20 \div 2 - 0 \times 1 + 2 \times 5$?

p2. Today is Saturday, January $25, 2020$. Exactly four hundred years from today, January $25, 2420$, is again a Saturday. How many weekend days (Saturdays and Sundays) are in February, 2420? (January has 31 days and in year 2040, February has 29 days.)

p3. Given that there are four people sitting around a circular table, and two of them stand up, what is the probability that the two of them were originally sitting next to each other?

p4. What is the area of a triangle with side lengths 5, 5, and 6?

p5. Six people go to OBA Noodles on Main Street. Each person has $1/2$ probability to order Duck Noodle Soup, $1/3$ probability to order OBA Ramen, and $1/6$ probability to order Kimchi Udon Soup. What is the probability that three people get Duck Noodle Soup, two people get OBA Ramen, and one person gets Kimchi Udon Soup?

p6. Among all positive integers a and b that satisfy $a^b = 64$, what is the minimum possible value of $a + b$?

p7. A positive integer n is called trivial if its tens digit divides n. How many two-digit trivial numbers are there?

p8. Triangle ABC has $AB = 5$, $BC = 13$, and $AC = 12$. Square $BCDE$ is constructed outside of the triangle. The perpendicular line from A to side DE cuts the square into two parts. What is the positive difference in their areas?

p9. In an increasing arithmetic sequence, the first, third, and ninth terms form an increasing geometric sequence (in that order). Given that the first term is 5, find the sum of the first nine terms of the arithmetic sequence.

p10. Square $ABCD$ has side length 1. Let points C' and D' be the reflections of points C and

D over lines AB and BC, respectively. Let P be the center of square $ABCD$. What is the area of the concave quadrilateral $PD'BC'$?

p11. How many four-digit palindromes are multiples of 7? (A palindrome is a number which reads the same forwards and backwards.)

p12. Let A and B be positive integers such that the absolute value of the difference between the sum of the digits of A and the sum of the digits of $(A + B)$ is 14. What is the minimum possible value for B?

p13. Clark writes the following set of congruences: $x \equiv a$ (mod 6), $x \equiv b$ (mod 10), $x \equiv c$ (mod 15), and he picks a, b, and c to be three randomly chosen integers. What is the probability that a solution for x exists?

p14. Vincent the bug is crawling on the real number line starting from 2020. Each second, he may crawl from x to $x - 1$, or teleport from x to $\frac{x}{3}$. What is the least number of seconds needed for Vincent to get to 0?

p15. How many positive divisors of 2020 do not also divide 1010?

p16. A bishop is a piece in the game of chess that can move in any direction along a diagonal on which it stands. Two bishops attack each other if the two bishops lie on the same diagonal of a chessboard. Find the maximum number of bishops that can be placed on an 8×8 chessboard such that no two bishops attack each other.

p17. Let ABC be a right triangle with hypotenuse 20 and perimeter 41. What is the area of ABC?

p18. What is the remainder when $x^{19} + 2x^{18} + 3x^{17} + ... + 20$ is divided by $x^2 + 1$?

p19. Ben splits the integers from 1 to 1000 into 50 groups of 20 consecutive integers each, starting with $\{1, 2, ..., 20\}$. How many of these groups contain at least one perfect square?

p20. Trapezoid $ABCD$ with AB parallel to CD has $AB = 10$, $BC = 20$, $CD = 35$, and $AD = 15$. Let AD and BC intersect at P and let AC and BD intersect at Q. Line PQ intersects AB at R. What is the length of AR?

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