Exeter Math Club Competition 2022

www.artofproblemsolving.com/community/c3003285

by parmenides51

– Team Round

– **p1.** Compute $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55$.

p2. Given that a, b, and c are positive integers such that $a + b = 9$ and $bc = 30$, find the minimum possible value of $a + c$.

p3. Points X and Y lie outside regular pentagon $ABCDE$ such that ABX and DEY are equilateral triangles. Find the degree measure of $\angle XCY$.

p4. Let N be the product of the positive integer divisors of $8!$, including itself. The largest integer power of 2 that divides N is 2^k. Compute k.

p5. Let $A = (-20, 22)$, $B = (k, 0)$, and $C = (202, 2)$ be points on the coordinate plane. Given that $\angle ABC = 90^o$, find the sum of all possible values of k.

p6. Tej is typing a string of Ls and Os that consists of exactly 7 Ls and 4 Os. How many different strings can he type that do not contain the substring 'LOL' anywhere? A substring is a sequence of consecutive letters contained within the original string.

p7. How many ordered triples of integers (a, b, c) satisfy both $a+b-c = 12$ and $a^2+b^2-c^2 = 24$?

p8. For how many three-digit base-7 numbers $\overline{ABC}_7$ does $\overline{ABC}_7$ divide $\overline{ABC}_{10}$? (Note: $\overline{ABC}_D$ refers to the number whose digits in base D are, from left to right, A, B, and C; for example, $\overline{123}_4$ equals 27 in base ten).

p9. Natasha is sitting on one of the 35 squares of a 5-by-7 grid of squares. Wanda wants to walk through every square on the board exactly once except the one Natasha is on, starting and ending on any 2 squares she chooses, such that from any square she can only go to an adjacent square (two squares are adjacent if they share an edge). How many squares can Natasha choose to sit on such that Wanda cannot go on her walk?

p10. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Point P lies inside ABC and points D, E, and F lie on sides BC, CA, and AB, respectively, so that $PD \perp BC$, $PE \perp CA$, and $PF \perp AB$. Given that PD, PE, and PF are all integers, find the sum of all possible distinct values of $PD \cdot PE \cdot PF$.

p11. A palindrome is a positive integer which is the same when read forwards or backwards. Find the sum of the two smallest palindromes that are multiples of 137.

p12. Let $P(x) = x^2+px+q$ be a quadratic polynomial with positive integer coefficients. Compute the least possible value of p such that 220 divides p and the equation $P(x^3) = P(x)$ has at least four distinct integer solutions.

p13. Everyone at a math club is either a truth-teller, a liar, or a piggybacker. A truth-teller always tells the truth, a liar always lies, and a piggybacker will answer in the style of the previous person who spoke (i.e., if the person before told the truth, they will tell the truth, and if the person before lied, then they will lie). If a piggybacker is the first one to talk, they will randomly either tell the truth or lie. Four seniors in the math club were interviewed and here was their conversation:

Neil: There are two liars among us.
Lucy: Neil is a piggybacker.
Kevin: Excluding me, there are more truth-tellers than liars here.
Neil: Actually, there are more liars than truth-tellers if we exclude Kevin.
Jacob: One plus one equals three.

Define the base-4 number $M = \overline{NLKJ}_4$, where each digit is 1 for a truth-teller, 2 for a piggybacker, and 3 for a liar (N corresponds to Neil, L to Lucy, K corresponds to Kevin, and J corresponds to Jacob). What is the sum of all possible values of M, expressed in base 10?

p14. An equilateral triangle of side length 8 is tiled by 64 equilateral triangles of unit side length to form a triangular grid. Initially, each triangular cell is either living or dead. The grid evolves over time under the following rule: every minute, if a dead cell is edge-adjacent to at least two living cells, then that cell becomes living, and any living cell remains living. Given that every cell in the grid eventually evolves to be living, what is the minimum possible number of living cells in the initial grid?

p15. In triangle ABC, $AB = 7$, $BC = 11$, and $CA = 13$. Let Γ be the circumcircle of ABC and let M, N, and P be the midpoints of minor arcs BC , CA, and AB of Γ, respectively. Given that K denotes the area of ABC and L denotes the area of the intersection of ABC and MNP, the ratio L/K can be written as a/b , where a and b are relatively prime positive integers. Compute $a + b$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Guts Round

– Round 1

p1. Let $ABCDEF$ be a regular hexagon. How many acute triangles have all their vertices among the vertices of $ABCDEF$?

p2. A rectangle has a diagonal of length 20. If the width of the rectangle is doubled, the length of the diagonal becomes 22. Given that the width of the original rectangle is w, compute w^2.

p3. The number $\overline{2022A20B22}$ is divisible by 99. What is $A + B$?

Round 2

p4. How many two-digit positive integers have digits that sum to at least 16?

p5. For how many integers k less than 10 do there exist positive integers x and y such that $k = x^2 - xy + y^2$?

p6. Isosceles trapezoid $ABCD$ is inscribed in a circle of radius 2 with $AB \parallel CD$, $AB = 2$, and one of the interior angles of the trapezoid equal to 110^o. What is the degree measure of minor arc CD?

Round 3

p7. In rectangle $ALEX$, point U lies on side EX so that $\angle AUL = 90^o$. Suppose that $UE = 2$ and $UX = 12$. Compute the square of the area of $ALEX$.

p8. How many digits does 20^{22} have?

p9. Compute the units digit of $3 + 3^3 + 3^{3^3} + ... + 3^{3^{\cdot^{\cdot^{\cdot^3}}}}$, where the last term of the series has 2022 3s.

Round 4

p10. Given that $\sqrt{x-1}+\sqrt{x}=\sqrt{x+1}$ for some real number x, the number x^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

p11. Eric the Chicken Farmer arranges his 9 chickens in a 3-by-3 grid, with each chicken being exactly one meter away from its closest neighbors. At the sound of a whistle, each chicken simultaneously chooses one of its closest neighbors at random and moves $\frac{1}{2}$ of a unit towards it. Given that the expected number of pairs of chickens that meet can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers, compute $p+q$.

p12. For a positive integer n, let $s(n)$ denote the sum of the digits of n in base 10. Find the greatest positive integer n less than 2022 such that $s(n)=s(n^2)$.

PS. You should use hide for answers. Rounds 5-8 have been posted here (`https://artofproblemsolving.com/community/c3h2949432p26408285`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Find the number of six-digit positive integers that satisfy all of the following conditions:
(i) Each digit does not exceed 3.
(ii) The number 1 cannot appear in two consecutive digits.
(iii) The number 2 cannot appear in two consecutive digits.

p14. Find the sum of all distinct prime factors of 103040301.

p15. Let $ABCA'B'C'$ be a triangular prism with height 3 where bases ABC and $A'B'C'$ are equilateral triangles with side length $\sqrt{6}$. Points P and Q lie inside the prism so that $ABCP$ and $A'B'C'Q$ are regular tetrahedra. The volume of the intersection of these two tetrahedra can be expressed in the form $\frac{\sqrt{m}}{n}$, where m and n are positive integers and m is not divisible by the square of any prime. Find $m+n$.

Round 6

p16. Let $a_0, a_1, ...$ be an infinite sequence such that $a_n^2-a_{n-1}a_{n+1}=a_n-a_{n-1}$ for all positive integers n. Given that $a_0 = 1$ and $a_1 = 4$, compute the smallest positive integer k such that a_k is an integer multiple of 220.

p17. Vincent the Bug is on an infinitely long number line. Every minute, he jumps either 2 units to the right with probability $\frac{2}{3}$ or 3 units to the right with probability $\frac{1}{3}$. The probability that Vincent never lands exactly 15 units from where he started can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is $p + q$?

p18. Battler and Beatrice are playing the "Octopus Game." There are 2022 boxes lined up in a row, and inside one of the boxes is an octopus. Beatrice knows the location of the octopus, but Battler does not. Each turn, Battler guesses one of the boxes, and Beatrice reveals whether or not the octopus is contained in that box at that time. Between turns, the octopus teleports to an adjacent box and secretly communicates to Beatrice where it teleported to. Find the least positive integer B such that Battler has a strategy to guarantee that he chooses the box containing the octopus in at most B guesses.

Round 7

p19. Given that $f(x) = x^2 - 2$ the number $f(f(f(f(f(f(f(2.5)))))))$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Find the greatest positive integer n such that 2^n divides $ab + a + b - 1$.

p20. In triangle ABC, the shortest distance between a point on the A-excircle ω and a point on the B-excircle Ω is 2. Given that $AB = 5$, the sum of the circumferences of ω and Ω can be written in the form $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. What is $m + n$? (Note: The A-excircle is defined to be the circle outside triangle ABC that is tangent to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and to the side BC. The B-excircle is defined similarly for vertex B.)

p21. Let $a_0, a_1, ...$ be an infinite sequence such that $a_0 = 1$, $a_1 = 1$, and there exists two fixed integer constants x and y for which a_{n+2} is the remainder when $xa_{n+1} + ya_n$ is divided by 15 for all nonnegative integers n. Let t be the least positive integer such that $a_t = 1$ and $a_{t+1} = 1$ if such an integer exists, and let $t = 0$ if such an integer does not exist. Find the maximal value of t over all possible ordered pairs (x, y).

Round 8

p22. A mystic square is a 3 by 3 grid of distinct positive integers such that the least common multiples of the numbers in each row and column are the same. Let M be the least possible maximal element in a mystic square and let N be the number of mystic squares with M as their maximal element. Find $M + N$.

p23. In triangle ABC, $AB = 27$, $BC = 23$, and $CA = 34$. Let X and Y be points on sides AB and AC, respectively, such that $BX = 16$ and $CY = 7$. Given that O is the circumcenter of BXY , the value of CO^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

p24. Alan rolls ten standard fair six-sided dice, and multiplies together the ten numbers he obtains. Given that the probability that Alan's result is a perfect square is $\frac{a}{b}$, where a and b are relatively prime positive integers, compute a.

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c3h2949416p26408251). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Accuracy

– **p1.** At a certain point in time, 20% of seniors, 30% of juniors, and 50% of sophomores at a school had a cold. If the number of sick students was the same for each grade, the fraction of sick students across all three grades can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

p2. The average score on Mr. Feng's recent test is a 63 out of 100. After two students drop out of the class, the average score of the remaining students on that test is now a 72. What is the maximum number of students that could initially have been in Mr. Feng's class? (All of the scores on the test are integers between 0 and 100, inclusive.)

p3. Madeline is climbing Celeste Mountain. She starts at $(0, 0)$ on the coordinate plane and wants to reach the summit at $(7, 4)$. Every hour, she moves either 1 unit up or 1 unit to the right. A strawberry is located at each of $(1, 1)$ and $(4, 3)$. How many paths can Madeline take so that she encounters exactly one strawberry?

p4. Let E be a point on side AD of rectangle $ABCD$. Given that $AB = 3$, $AE = 4$, and $\angle BEC = \angle CED$, the length of segment CE can be written as $\sqrt{a}$ for some positive integer a. Find a.

p5. Lucy has some spare change. If she were to convert it into quarters and pennies, the minimum number of coins she would need is 66. If she were to convert it into dimes and pennies, the minimum number of coins she would need is 147. How much money, in cents, does Lucy have?

p6. For how many positive integers x does there exist a triangle with altitudes of length 20, 22, and x?

p7. Compute the number of positive integers x for which $\frac{x^{20}}{x+22}$ is an integer.

p8. Vincent the Bug is crawling along an octagonal prism. He starts on a fixed vertex A, visits all other vertices exactly once by traveling along the edges, and returns to A. Find the number of paths Vincent could have taken.

p9. Point U is chosen inside square $ALEX$ so that $\angle AUL = 90^o$. Given that $UL = 56$ and $UE = 65$, what is the sum of all possible values for the area of square $ALEX$?

p10. Miranda has prepared 8 outfits, no two of which are the same quality. She asks her intern Andrea to order these outfits for the new runway show. Andrea first randomly orders the outfits in a list. She then starts removing outfits according to the following method: she chooses a random outfit which is both immediately preceded and immediately succeeded by a better outfit and then removes it. Andrea repeats this process until there are no outfits that can be removed. Given that the expected number of outfits in the final routine can be written as $\frac{a}{b}$ for some relatively prime positive integers a and b, find $a + b$.

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– Individual Speed

– *20 problems for 25 minutes.*

p1. Compute $(2 + 0)(2 + 2)(2 + 0)(2 + 2)$.

p2. Given that 25% of x is 120% of 30% of 200, find x.

p3. Jacob had taken a nap. Given that he fell asleep at $4 : 30$ PM and woke up at $6 : 23$ PM later that same day, for how many minutes was he asleep?

p4. Kevin is painting a cardboard cube with side length 12 meters. Given that he needs exactly one can of paint to cover the surface of a rectangular prism that is 2 meters long, 3 meters wide, and 6 meters tall, how many cans of paint does he need to paint the surface of his cube?

p5. How many nonzero digits does $200 \times 25 \times 8 \times 125 \times 3$ have?

p6. Given two real numbers x and y, define $x\#y = xy + 7x - y$. Compute the absolute value of $0\#(1\#(2\#(3\#4)))$.

p7. A 3-by-5 rectangle is partitioned into several squares of integer side length. What is the fewest number of such squares? Squares in this partition must not overlap and must be contained within the rectangle.

p8. Points A and B lie in the plane so that $AB = 24$. Given that C is the midpoint of AB, D is the midpoint of BC, E is the midpoint of AD, and F is the midpoint of BD, find the length of segment EF.

p9. Vincent the Bug and Achyuta the Anteater are climbing an infinitely tall vertical bamboo stalk. Achyuta begins at the bottom of the stalk and climbs up at a rate of 5 inches per second, while Vincent begins somewhere along the length of the stalk and climbs up at a rate of 3 inches per second. After climbing for t seconds, Achyuta is half as high above the ground as Vincent. Given that Achyuta catches up to Vincent after another 160 seconds, compute t.

p10. What is the minimum possible value of $|x - 2022| + |x - 20|$ over all real numbers x?

p11. Let $ABCD$ be a rectangle. Lines ℓ_1 and ℓ_2 divide $ABCD$ into four regions such that ℓ_1 is parallel to AB and line ℓ_2 is parallel to AD. Given that three of the regions have area 6, 8, and 12, compute the sum of all possible areas of the fourth region.

p12. A diverse number is a positive integer that has two or more distinct prime factors. How many diverse numbers are less than 50?

p13. Let x, y, and z be real numbers so that $(x + y)(y + z) = 36$ and $(x + z)(x + y) = 4$. Compute $y^2 - x^2$.

p14. What is the remainder when $1^{10} + 3^{10} + 7^{10}$ is divided by 58?

p15. Let $A = (0, 1)$, $B = (3, 5)$, $C = (1, 4)$, and $D = (3, 4)$ be four points in the plane. Find the minimum possible value of $AP + BP + CP + DP$ over all points P in the plane.

p16. In trapezoid $ABCD$, points E and F lie on sides BC and AD, respectively, such that $AB \parallel CD \parallel EF$. Given that $AB = 3$, $EF = 5$, and $CD = 6$, the ratio $\frac{[ABEF]}{[CDFE]}$ can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$. (Note: $[F]$ denotes the area of F.)

p17. For sets X and Y , let $|X \cap Y|$ denote the number of elements in both X and Y and $|X \cup Y|$ denote the number of elements in at least one of X or Y . How many ordered pairs of subsets (A, B) of $\{1, 2, 3, ..., 8\}$ are there such that $|A \cap B| = 2$ and $|A \cup B| = 5$?

p18. A tetromino is a polygon composed of four unit squares connected orthogonally (that is, sharing a edge). A tri-tetromino is a polygon formed by three orthogonally connected tetrominoes. What is the maximum possible perimeter of a tri-tetromino?

p19. The numbers from 1 through 2022, inclusive, are written on a whiteboard. Every day, Hermione erases two numbers a and b and replaces them with $ab+a+b$. After some number of days, there is only one number N remaining on the whiteboard. If N has k trailing nines in its decimal representation, what is the maximum possible value of k?

p20. Evaluate $5(2^2 + 3^2) + 7(3^2 + 4^2) + 9(4^2 + 5^2) + ... + 199(99^2 + 100^2)$.

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