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– Day 0

1 For a positive integer n, consider a square cake which is divided into $n \times n$ pieces with at most one strawberry on each piece. We say that such a cake is *delicious* if both diagonals are fully occupied, and each row and each column has an odd number of strawberries.
Find all positive integers n such that there is an $n \times n$ delicious cake with exactly $\left\lceil \frac{n^2}{2} \right\rceil$ strawberries on it.

2 Prove that, for all positive integers m and n, we have

$$\left\lfloor m\sqrt{2} \right\rfloor \cdot \left\lfloor n\sqrt{7} \right\rfloor < \left\lfloor mn\sqrt{14} \right\rfloor.$$

3 Let P be a point on the circumcircle of acute triangle ABC. Let D, E, F be the reflections of P in the A-midline, B-midline, and C-midline. Let ω be the circumcircle of the triangle formed by the perpendicular bisectors of AD, BE, CF.

Show that the circumcircles of $\triangle ADP, \triangle BEP, \triangle CFP$, and ω share a common point.

– Day 1

1 Given a positive integer k show that there exists a prime p such that one can choose distinct integers $a_1, a_2 \cdots, a_{k+3} \in \{1, 2, \cdots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \cdots, k$.

South Africa

2 Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Israel

3 Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \dots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \dots, n\}$ that are red.

Prove that there exists a positive integer $a \in \{1, 2, \dots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \dots, N$.

Netherlands

– Day 2

1 Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90$, $CDA > 90$ and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

Slovakia

2 For any odd prime p and any integer n, let $d_p(n) \in \{0, 1, \dots, p-1\}$ denote the remainder when n is divided by p. We say that $(a_0, a_1, a_2, \dots)$ is a *p-sequence*, if a_0 is a positive integer coprime to p, and $a_{n+1} = a_n + d_p(a_n)$ for $n \geqslant 0$.
(a) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ such that $a_n > b_n$ for infinitely many n, and $b_n > a_n$ for infinitely many n?
(b) Do there exist infinitely many primes p for which there exist p-sequences $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ such that $a_0 < b_0$, but $a_n > b_n$ for all $n \geqslant 1$?

United Kingdom

3 Consider any rectangular table having finitely many rows and columns, with a real number $a(r, c)$ in the cell in row r and column c. A pair (R, C), where R is a set of rows and C a set of columns, is called a *saddle pair* if the following two conditions are satisfied:

- (i) For each row r', there is $r \in R$ such that $a(r, c) \geqslant a(r', c)$ for all $c \in C$;
- (ii) For each column c', there is $c \in C$ such that $a(r, c) \leqslant a(r, c')$ for all $r \in R$.

A saddle pair (R, C) is called a *minimal pair* if for each saddle pair (R', C') with $R' \subseteq R$ and $C' \subseteq C$, we have $R' = R$ and $C' = C$. Prove that any two minimal pairs contain the same number of rows.

– Day 3

1 *Version 1*. Let n be a positive integer, and set $N = 2^n$. Determine the smallest real number a_n such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Version 2. For every positive integer N, determine the smallest real number b_N such that, for all real x,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

2 In the plane, there are $n \geqslant 6$ pairwise disjoint disks $D_1, D_2, \ldots, D_n$ with radii $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$. For every $i = 1, 2, \ldots, n$, a point P_i is chosen in disk D_i. Let O be an arbitrary point in the plane. Prove that

$$OP_1 + OP_2 + \ldots + OP_n \geqslant R_6 + R_7 + \ldots + R_n.$$

(A disk is assumed to contain its boundary.)

3 Determine all functions f defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

- (i) $f(n) \neq 0$ for at least one n;
- (ii) $f(xy) = f(x) + f(y)$ for every positive integers x and y;
- (iii) there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

– Day 4

1 For each prime p, construct a graph G_p on $\{1, 2, \ldots p\}$, where $m \neq n$ are adjacent if and only if p divides $(m^2 + 1 - n)(n^2 + 1 - m)$. Prove that G_p is disconnected for infinitely many p

2 Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on AB, BC, CD, DA such that $KLMN$ is a rhombus with $KL \parallel AC$ and $LM \parallel BD$. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of $\triangle ANK, \triangle BKL, \triangle CLM, \triangle DMN$.

Prove that the common internal tangents to ω_A, and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

3 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{a^2+b^2}(a+b) = af(a) + bf(b)$$

for all integers a and b

– Day 5

1 In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \ldots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- the quadrilaterals $Q_1, \ldots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral Q_i has three corners of one color and one corner of the other color.

2 Let $\mathcal{A}$ be the set of all $n \in \mathbb{N}$ for which there exist $k \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{k-1} \in \{1, 2, \ldots, 9\}$ such that $a_0 \geq a_1 \geq \cdots \geq a_{k-1}$ and $n = a_0 + a_1 \cdot 10^1 + \cdots + a_{k-1} \cdot 10^{k-1}$. Let $\mathcal{B}$ be the set of all $m \in \mathbb{N}$

for which there exist $l \in \mathbb{N}$ and $b_0, b_1, \dots, b_{l-1} \in \{1, 2, \dots, 9\}$ such that $b_0 \le b_1 \le \dots \le b_{l-1}$ and $m = b_0 + b_1 \cdot 10^1 + \dots + b_{l-1} \cdot 10^{l-1}$.

- Are there infinitely many $n \in \mathcal{A}$ such that $n^2 - 3 \in \mathcal{A}$?
- Are there infinitely many $m \in \mathcal{B}$ such that $m^2 - 3 \in \mathcal{B}$?

Proposed by Pakawut Jiradilok and Wijit Yangjit

3 A magician intends to perform the following trick. She announces a positive integer n, along with $2n$ real numbers $x_1 < \dots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \dots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience. Can the magician find a strategy to perform such a trick?

– Day 6

1 Let $\mathcal{A}$ denote the set of all polynomials in three variables x, y, z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x + y + z)P(x, y, z) + (xy + yz + zx)Q(x, y, z) + xyzR(x, y, z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest non-negative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all non-negative integers i, j, k satisfying $i + j + k \ge n$.

2 The Fibonacci numbers $F_0, F_1, F_2, \dots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \ge 1$. Given an integer $n \ge 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \dots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.

Proposed by Croatia

3 Let ABC be a triangle with $AB < AC$, incenter I, and A excenter I_A. The incircle meets BC at D. Define $E = AD \cap BI_A$, $F = AD \cap CI_A$. Show that the circumcircle of $\triangle AID$ and $\triangle I_A EF$ are tangent to each other