VMO 2022

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by parmenides51, wardtnt1234

– Day 1

1 Let a be a non-negative real number and a sequence (u_n) defined as: $u_1=6, u_{n+1}=\frac{2n+a}{n}+\sqrt{\frac{n+a}{n}u_n+4}, \forall n\ge 1$
a) With $a=0$, prove that there exist a finite limit of (u_n) and find that limit
b) With $a\ge 0$, prove that there exist a finite limit of (u_n)

2 Find all function $f:\mathbb{R}^+\to\mathbb{R}^+$ such that:

$$f\left(\frac{f(x)}{x}+y\right)=1+f(y),\quad \forall x,y\in\mathbb{R}^+.$$

3 Let ABC be a triangle. Point E,F moves on the opposite ray of BA,CA such that $BF=CE$. Let M,N be the midpoint of BE,CF. BF cuts CE at D
a) Suppost that I is the circumcenter of (DBE) and J is the circumcenter of (DCF), Prove that $MN \parallel IJ$
b) Let K be the midpoint of MN and H be the orthocenter of triangle AEF. Prove that when E varies on the opposite ray of BA, HK go through a fixed point

4 For every pair of positive integers (n,m) with $n<m$, denote $s(n,m)$ be the number of positive integers such that the number is in the range $[n,m]$ and the number is coprime with m. Find all positive integers $m\ge 2$ such that m satisfy these condition:
i) $\frac{s(n,m)}{m-n}\ge\frac{s(1,m)}{m}$ for all $n=1,2,...,m-1$;
ii) 2022^m+1 is divisible by m^2

– Day 2

1 Consider 2 non-constant polynomials $P(x),Q(x)$, with nonnegative coefficients. The coefficients of $P(x)$ is not larger than 2021 and $Q(x)$ has at least one coefficient larger than 2021. Assume that $P(2022)=Q(2022)$ and $P(x),Q(x)$ has a root $\frac{p}{q}\ne 0 (p,q\in\mathbb{Z},(p,q)=1)$. Prove that $|p|+n|q|\le Q(n)-P(n)$ for all $n=1,2,...,2021$

2 We are given 4 similar dices. Denote $x_i (1\le x_i\le 6)$ be the number of dots on a face appearing on the i-th dice $1\le i\le 4$
a) Find the numbers of (x_1,x_2,x_3,x_4)
b) Find the probability that there is a number x_j such that x_j is equal to the sum of the other 3

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numbers
c) Find the probability that we can divide x_1, x_2, x_3, x_4 into 2 groups has the same sum

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3 Let ABC be an acute triangle, B, C fixed, A moves on the big arc BC of (ABC). Let O be the circumcenter of (ABC) $(B, O, C$ are not collinear, $AB \neq AC)$, (I) is the incircle of triangle ABC. (I) tangents to BC at D. Let I_a be the A-excenter of triangle ABC. I_aD cuts OI at L. Let E lies on (I) such that $DE \parallel AI$.
a) LE cuts AI at F. Prove that $AF = AI$.
b) Let M lies on the circle (J) go through I_a, B, C such that $I_aM \parallel AD$. MD cuts (J) again at N. Prove that the midpoint T of MN lies on a fixed circle.
