Sharygin Geometry Olympiad 2022

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– First (Correspondence) Round

1 Let O and H be the circumcenter and the orthocenter respectively of triangle ABC. Itis known that BH is the bisector of angle ABO. The line passing through O and parallel to AB meets AC at K. Prove that $AH = AK$

2 Let $ABCD$ be a curcumscribed quadrilateral with incenter I, and let O_1, O_2 be the circumcenters of triangles AID and CID. Prove that the circumcenter of triangle O_1IO_2 lies on the bisector of angle ABC

3 Let CD be an altitude of right-angled triangle ABC with $\angle C = 90$. Regular trianglesAED and CFD are such that E lies on the same side from AB as C, and F lies on the same side from CD as B. The line EF meets AC at L. Prove that $FL = CL + LD$

4 Let AA_1, BB_1, CC_1 be the altitudes of acute angled triangle ABC. A_2 be the touching point of the incircle of triangle AB_1C_1 with B_1C_1, points B_2, C_2 be defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

5 Let the diagonals of cyclic quadrilateral $ABCD$ meet at point P. The line passing through P and perpendicular to PD meets AD at point D_1, a point A_1 is defined similarly. Prove that the tangent at P to the circumcircle of triangle D_1PA_1 is parallel to BC.

6 The incircle and the excircle of triangle ABC touch the side AC at points P and Q respectively. The lines BP and BQ meet the circumcircle of triangle ABC for the second time at points P' and Q' respectively.
Prove that

$$PP' > QQ'$$

7 A square with center F was constructed on the side AC of triangle ABC outside it. After this, everything was erased except F and the midpoints N, K of sides BC, AB.
Restore the triangle.

8 Points P, Q, R lie on the sides AB, BC, CA of triangle ABC in such a way that $AP = PR, CQ = QR$. Let H be the orthocenter of triangle PQR, and O be the circumcenter of triangle ABC.

Prove that
$$OH||AC$$
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9 The sides AB, BC, CD and DA of quadrilateral $ABCD$ touch a circle with center I at points K, L, M and N respectively. Let P be an arbitrary point of line AI. Let PK meet BI at point Q, QL meet CI at point R, and RM meet DI at point S.
Prove that P, N and S are collinear.

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10 Let ω_1 be the circumcircle of triangle ABC and O be its circumcenter. A circle ω_2 touches the sides AB, AC, and touches the arc BC of ω_1 at point K. Let I be the incenter of ABC.
Prove that the line OI contains the symmedian of triangle AIK.

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11 Let ABC be a triangle with $\angle A = 60^o$ and T be a point such that $\angle ATB = \angle BTC = \angle ATC$. A circle passing through B, C and T meets AB and AC for the second time at points K and L.Prove that the distances from K and L to AT are equal.

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12 Let K, L, M, N be the midpoints of sides BC, CD, DA, AB respectively of a convex quadrilateral $ABCD$. The common points of segments AK, BL, CM, DN divide each of them into three parts. It is known that the ratio of the length of the medial part to the length of the whole segment is the same for all segments. Does this yield that $ABCD$ is a parallelogram?

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13 Eight points in a general position are given in the plane. The areas of all 56 triangles with vertices at these points are written in a row. Prove that it is possible to insert the symbols "$+$" and "$-$" between them in such a way
that the obtained sum is equal to zero.

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14 A triangle ABC is given. Let C' and C'_a be the touching points of sideline AB with the incircle and with the excircle touching the side BC. Points C'_b, C'_c, A', A'_a, A'_b, A'_c, B', B'_a, B'_b, B'_c are defined similarly. Consider the lengths of 12 altitudes of triangles $A'B'C'$, $A'_aB'_aC'_a$, $A'_bB'_bC'_b$, $A'_cB'_cC'_c$.
(a) (8-9) Find the maximal number of different lengths.
(b) (10-11) Find all possible numbers of different lengths.

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15 A line l parallel to the side BC of triangle ABC touches its incircle and meets its circumcircle at points D and E. Let I be the incenter of ABC. Prove that $AI^2 = AD \cdot AE$.

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16 Let $ABCD$ be a cyclic quadrilateral, $E = AC \cap BD$, $F = AD \cap BC$. The bisectors of angles AFB and AEB meet CD at points X, Y . Prove that A, B, X, Y are concyclic.

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17 Let a point P lie inside a triangle ABC. The rays starting at P and crossing the sides BC, AC, AB under the right angle meet the circumcircle of ABC at A_1, B_1, C_1 respectively. It is known that lines AA_1, BB_1, CC_1 concur at point Q. Prove that all such lines PQ concur.

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18 The products of the opposite sidelengths of a cyclic quadrilateral $ABCD$ are equal. Let B' be the reflection of B about AC. Prove that the circle passing through A, B', D touches AC

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19 Let I be the incenter of triangle ABC, and K be the common point of BC with the external bisector of angle A. The line KI meets the external bisectors of angles B and C at points X and Y . Prove that $\angle BAX = \angle CAY$

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20 Let O, I be the circumcenter and the incenter of $\triangle ABC$; R,r be the circumradius and the inradius; D be the touching point of the incircle with BC; and N be an arbitrary point of segment ID. The perpendicular to ID at N meets the circumcircle of ABC at points X and Y . Let O_1 be the circumcircle of $\triangle XIY$.

Find the product $OO_1 \cdot IN$.

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21 The circumcenter O, the incenter I, and the midpoint M of a diagonal of a bicentral quadrilateral were marked. After this the quadrilateral was erased. Restore it.

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22 Chords A_1A_2, A_3A_4, A_5A_6 of a circle Ω concur at point O. Let B_i be the second common point of Ω and the circle with diameter OA_i . Prove that chords B_1B_2, B_3B_4, B_5B_6 concur.

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23 An ellipse with focus F is given. Two perpendicular lines passing through F meet the ellipse at four points. The tangents to the ellipse at these points form a quadrilateral circumscribed around the ellipse. Prove that this quadrilateral is inscribed into a conic with focus F

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24 Let $OABCDEF$ be a hexagonal pyramid with base $ABCDEF$ circumscribed around a sphere ω. The plane passing through the touching points of ω with faces OFA, OAB and $ABCDEF$ meets OA at point A_1, points B_1, C_1, D_1, E_1 and F_1 are defined similarly. Let ℓ, m and n be the lines A_1D_1, B_1E_1 and C_1F_1 respectively. It is known that ℓ and m are coplanar, also m and n are coplanar. Prove that ℓ and n are coplanar.

– Final Round

8.1 Let $ABCD$ be a convex quadrilateral with $\angle BAD = 2\angle BCD$ and $AB = AD$. Let P be a point such that $ABCP$ is a parallelogram. Prove that $CP = DP$.

8.2 Let $ABCD$ be a right-angled trapezoid and M be the midpoint of its greater lateral side CD. Circumcircles ω_1 and ω_2 of triangles BCM and AMD meet for the second time at point E. Let ED meet ω_1 at point F, and FB meet AD at point G. Prove that GM bisects angle BGD.

8.3 A circle ω and a point P not lying on it are given. Let ABC be an arbitrary equilateral triangle inscribed into ω and A', B', C' be the projections of P to BC, CA, AB. Find the locus of centroids of triangles $A'B'C'$.

8.4 Let $ABCD$ be a cyclic quadrilateral, O be its circumcenter, P be a common points of its diagonals, and M, N be the midpoints of AB and CD respectively. A circle OPM meets for the second time segments AP and BP at points A_1 and B_1 respectively and a circle OPN meets for the second time segments CP and DP at points C_1 and D_1 respectively. Prove that the areas of quadrilaterals AA_1B_1B and CC_1D_1D are equal.

8.5 An incircle of triangle ABC touches AB, BC, AC at points C_1, A_1,B_1 respectively. Let A' be the reflection of A_1 about B_1C_1, point C' is defined similarly. Lines $A'C_1$ and $C'A_1$ meet at point D. Prove that $BD \parallel AC$.

8.6 Two circles meeting at points A, B and a point O lying outside them are given. Using a compass and a ruler construct a ray with origin O meeting the first circle at point C and the second one at point D in such a way that the ratio $OC : OD$ be maximal.

8.7 Ten points on a plane a such that any four of them lie on the boundary of some square. Is obligatory true that all ten points lie on the boundary of some square?

8.8 An isosceles trapezoid $ABCD$ ($AB = CD$) is given. A point P on its circumcircle is such that segments CP and AD meet at point Q. Let L be tha midpoint ofQD. Prove that the diagonal of the trapezoid is not greater than the sum of distances from the midpoints of the lateral sides to ana arbitrary point of line PL.

9.1 Let BH be an altitude of right angled triangle ABC($\angle B = 90^o$). An excircle of triangle ABH opposite to B touches AB at point A_1; a point C_1 is defined similarly. Prove that $AC//A_1C_1$.

9.2 Let circles s_1 and s_2 meet at points A and B. Consider all lines passing through A and meeting the circles for the second time at points P_1 and P_2 respectively. Construct by a compass and a ruler a line such that $AP_1.AP_2$ is maximal.

9.3 A medial line parallel to the side AC of triangle ABC meets its circumcircle at points at X and Y. Let I be the incenter of triangle ABC and D be the midpoint of arc AC not containing B.A point L lie on segment DI in such a way that $DL = BI/2$. Prove that $\angle IXL = \angle IYL$.

9.4 Let ABC be an isosceles triangle with $AB = AC$, P be the midpoint of the minor arc AB of its circumcircle, and Q be the midpoint of AC. A circumcircle of triangle APQ centered at O meets AB for the second
time at point K. Prove that lines PO and KQ meet on the bisector of angle ABC.

9.5 Chords AB and CD of a circle ω meet at point E in such a way that $AD = AE = EB$. Let F be a point of segment CE such that $ED = CF$. The bisector of angle AFC meets an arc DAC at point P. Prove
that A, E, F, and P are concyclic.

9.6 Lateral sidelines AB and CD of a trapezoid $ABCD$ ($AD > BC$) meet at point P. Let Q be a point of segment AD such that $BQ = CQ$. Prove that the line passing through the circumcenters of triangles AQC and BQD is perpendicular to PQ.

9.7 Let H be the orthocenter of an acute-angled triangle ABC. The circumcircle of triangle AHC meets segments AB and BC at points P and Q. Lines PQ and AC meet at point R. A point K lies on the line PH
in such a way that $\angle KAC = 90^\circ$. Prove that KR is perpendicular to one of
the medians of triangle ABC.

9.8 Several circles are drawn on the plane and all points of their intersection or touching are marked. Is it possible that each circle contains exactly five marked points and each point belongs to exactly five circles?

10.1 $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two squares with their vertices arranged clockwise.The perpendicular bisector of segment $A_1B_1, A_2B_2, A_3B_3, A_4B_4$ and the perpendicular bisector of segment $A_2B_2, A_3B_3, A_4B_4, A_1B_1$ intersect at point P, Q, R, S respectively.Show that:$PR \perp QS$.

10.2 Let $ABCD$ be a convex quadrilateral. The common external tangents to circles (ABC) and (ACD) meet at point E, the common external tangents to circles (ABD) and (BCD) meet at point F. Let F lie on AC, prove that E lies on BD.

10.3 A line meets a segment AB at point C. Which is the maximal number of points X of this line such that one of angles AXC and BXC is equlal to a half of the second one?

10.4 Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D$. Prove that the midpoint of BD lies on the common internal tangent to the incircles of triangles ABC and ACD.

10.5 LetAB and AC be the tangents from a point A to a circle Ω. Let M be the midpoint of BC and P be an arbitrary point on this segment. A line AP meets Ω at points D and E. Prove that the common external tangents to circles MDP and MPE meet on the midline of triangle ABC.

10.6 Let O, I be the circumcenter and the incenter of triangle ABC, P be an arbitrary point on segment OI, P_A, P_B, and P_C be the second common points of lines PA, PB, and PC with the circumcircle of triangle ABC. Prove that the bisectors of angles BP_AC, CP_BA, and AP_CB concur at a point lying on OI.

10.7 Several circles are drawn on the plane and all points of their meeting or touching are marked. May be that each circle contains exactly four marked points and exactly four marked points lie on each circle?

10.8 Let $ABCA'B'C'$ be a centrosymmetric octahedron (vertices A and A', B and B', C and C' are opposite) such that the sums of four planar angles equal 240^o for each vertex. The Torricelli points T_1 and T_2 of triangles ABC and $A'BC$ are marked. Prove that the distances from T_1 and T_2 to BC are equal.