IMC 2016

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by j___d

–	Day 1
1	Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$. (a) Prove that $f(a)f(b) = 0$. (b) Give an example of such a function on $[0, 1]$. (Proposed by Alexandr Bolbot, Novosibirsk State University)
2	Let k and n be positive integers. A sequence $(A_1, \dots, A_k)$ of $n \times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_iA_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n. (Proposed by Fedor Petrov, St. Petersburg State University)
3	Let n be a positive integer. Also let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \dots, n$. Prove that $$\sum_{i=1}^{n} \frac{a_ib_i - b_i^2}{a_i + b_i} \leq \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n}(a_i + b_i)}$$. (Proposed by Daniel Strzelecki, Nicolaus Copernicus University in Toru, Poland)
4	Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties: (i) $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements; (ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$. Prove that $\mathcal{F}$ contains at least three sets with at least n elements. (Proposed by Fedor Petrov, St. Petersburg State University)
5	Let S_n denote the set of permutations of the sequence $(1, 2, \dots, n)$. For every permutation $\pi = (\pi_1, \dots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \leq i < j \leq n$ with $\pi_i > \pi_j$; i. e. the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n + 1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

(Proposed by Fedor Petrov, St. Petersburg State University)

– Day 2

1 Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \le 2.$$

(Proposed by Gerhard J. Woeginger, The Netherlands)

2 Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \ge |x-y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

(Proposed by Fedor Petrov, St. Petersburg State University)

3 Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo n. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x+1)+1)+1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Germany)

4 Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + \ldots + |x_k| \le n$. Prove that for every $n \ge 1$, we have $f(n-1)f(n+1) \le f(n)^2$.

(Proposed by Esteban Arreaga, Renan Finder and Jos Madrid, IMPA, Rio de Janeiro)

5 Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \le \frac{n}{\ln 2} \|A\|^{n-1}.$$

(Here $\|B\| = \sup_{\|x\| \le 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(Proposed by Ian Morris and Fedor Petrov, St. Petersburg State University)