Mid-Michigan Mathematical Olympiad, Grades 7-9

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by parmenides51

2002 **p1.** One out of 12 coins is counterfeited. It is known that its weight differs from the weight of a valid coin but it is unknown whether it is lighter or heavier. How to detect the counterfeited coin with the help of four trials using only a two-pan balance without weights?

p2. Below a 3-digit number cde is multiplied by a 2-digit number ab . Find all solutions a, b, c, d, e, f, g if it is known that they represent distinct digits.

		c	d	e
x			a	b
		f	e	g
+	c	d	e	
	b	b	c	g

p3. Find all integer n such that $\frac{n+1}{2n-1}$is an integer.

p4. There are several straight lines on the plane which split the plane in several pieces. Is it possible to paint the plane in brown and green such that each piece is painted one color and no pieces having a common side are painted the same color?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2003 **p1**. Is it possible to find n positive numbers such that their sum is equal to 1 and the sum of their squares is less than $\frac{1}{10}$?

p2. In the country of Sepulia, there are several towns with airports. Each town has a certain number of scheduled, round-trip connecting flights with other towns. Prove that there are two towns that have connecting flights with the same number of towns.

p3. A 4×4 magic square is a 4×4 table filled with numbers $1, 2, 3, ..., 16$ - with each number appearing exactly once - in such a way that the sum of the numbers in each row, in each column, and in each diagonal is the same. Is it possible to complete $\begin{bmatrix} 2 & 3 & * & * \\ 4 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ to a magic square? (That is, can you replace the stars with remaining numbers $1, 5, 6, ..., 16$, to obtain a magic square?)

p4. Is it possible to label the edges of a cube with the numbers $1, 2, 3, ..., 12$ in such a way that the sum of the numbers labelling the three edges coming into a vertex is the same for all vertices?

p5. (Bonus) Several ants are crawling along a circle with equal constant velocities (not necessarily in the same direction). If two ants collide, both immediately reverse direction and crawl with the same velocity. Prove that, no matter how many ants and what their initial positions are, they will, at some time, all simultaneously return to the initial positions.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2004 **p1.** Two players play the following game. On the lowest left square of an 8×8 chessboard there is a rook. The first player is allowed to move the rook up or to the right by an arbitrary number of squares. The second player is also allowed to move the rook up or to the right by an arbitrary number of squares. Then the first player is allowed to do this again, and so on. The one who moves the rook to the upper right square wins. Who has a winning strategy?

p2. In Crocodile Country there are banknotes of 1 dollar, 10 dollars, 100 dollars, and $1,000$ dollars. Is it possible to get 1,000,000 dollars by using $250,000$ banknotes?

p3. Fifteen positive numbers (not necessarily whole numbers) are placed around the circle. It is known that the sum of every four consecutive numbers is 30. Prove that each number is less than 15.

p4. Donald Duck has 100 sticks, each of which has length 1 cm or 3 cm. Prove that he can break into 2 pieces no more than one stick, after which he can compose a rectangle using all sticks.

p5. Three consecutive 2 digit numbers are written next to each other. It turns out that the resulting 6 digit number is divisible by 17. Find all such numbers.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2005 **p1.** Prove that no matter what digits are placed in the four empty boxes, the eight-digit number $9999\square\square\square\square$ is not a perfect square.

p2. Prove that the number $m/3 + m^2/2 + m^3/6$ is integral for all integral values of m.

p3. An elevator in a 100 store building has only two buttons: UP and DOWN. The UP button makes the elevator go 13 floors up, and the DOWN button makes it go 8 floors down. Is it possible to go from the 13th floor to the 8th floor?

p4. Cut the triangle shown in the picture into three pieces and rearrange them into a rectangle. (Pieces can not overlap.)
https://cdn.artofproblemsolving.com/attachments/4/b/ca707bf274ed54c1b22c4f65d3d0b0a5cfdc5
png

p5. Two players Tom and Sid play the following game. There are two piles of rocks, 7 rocks in the first pile and 9 rocks in the second pile. Each of the players in his turn can take either any amount of rocks from one pile or the same amount of rocks from both piles. The winner is the player who takes the last rock. Who does win in this game if Tom starts the game?

p6. In the next long multiplication example each letter encodes its own digit. Find these digits.

			a	b
*			c	d
		c	e	f
+		a	b	
	c	f	d	f

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2006 **p1.** Find all solutions a, b, c, d, e, f if it is known that they represent distinct digits and satisfy the following:

		a	b	c	a
+			d	d	e
				d	e
	d	f	f	d	d

p2. Explain whether it possible that the sum of two squares of positive whole numbers has all digits equal to 1:
$$n^2 + m^2 = 111...111$$

p3. Two players play the following game on an 8×8 chessboard. The first player can put a rook on an arbitrary square. Then the second player can put another rook on a free square that is not controlled by the first rook. Then the first player can put a new rook on a free square that is not controlled by the rooks on the board. Then the second player can do the same, etc. A player who cannot put a new rook on the board loses the game. Who has a winning strategy?

p4. Show that the difference $9^{2008} - 7^{2008}$ is divisible by 10.

p5. Is it possible to find distict positive whole numbers a, b, c, d, e such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1?$$

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2007 **p1.** The Evergreen School booked buses for a field trip. Altogether, 138 people went to West Lake, while 115 people went to East Lake. The buses all had the same number of seats and every bus has more than one seat. All seats were occupied and everybody had a seat. How many seats were on each bus?

p2. In New Scotland there are three kinds of coins: 1 cent, 6 cent, and 36 cent coins. Josh has 99 of the 36-cent coins (and no other coins). He is allowed to exchange a 36 cent coin for 6 coins of 6 cents, and to exchange a 6 cent coin for 6 coins of 1 cent. Is it possible that after several exchanges Josh will have 500 coins?

p3. Find all solutions a, b, c, d, e, f, g, h if these letters represent distinct digits and the following multiplication is correct:

		a	b	c
+			d	e
	f	a	g	c
x	b	b	h	
f	f	e	g	c

p4. Is it possible to find a rectangle of perimeter 10 m and cut it in rectangles (as many as you want) so that the sum of the perimeters is 500 m?

p5. The picture shows a maze with chambers (shown as circles) and passageways (shown as segments). A cat located in chamber C tries to catch a mouse that was originally in the chamber M. The cat makes the first move, moving from chamber C to one of the neighboring chambers. Then the mouse moves, then the cat, and so forth. At each step, the cat and the mouse can move to any neighboring chamber or not move at all. The cat catches the mouse by moving into the chamber currently occupied by the mouse. Can the cat get the mouse?
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png

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2008 **p1.** Jack made 3 quarts of fruit drink from orange and apple juice. His drink contains 45% of orange juice. Nick prefers more orange juice in the drink. How much orange juice should he add to the drink to obtain a drink composed of 60% of orange juice?

p2. A square is tiled by smaller squares as shown in the figure. Find the area of the black square in the middle if the perimeter of the big square $ABCD$ is 40 cm.
https://cdn.artofproblemsolving.com/attachments/8/c/d54925cba07f63ec8578048f46e1e730cb8df3.png

p3. For one particular number $a > 0$ the function f satisfies the equality $f(x+a) = \frac{1+f(x)}{1-f(x)}$ for all x. Show that f is a periodic function. (A function f is periodic with the period T if $f(x+T) = f(x)$ for any x.)

p4. If a, b, c, x, y, z are numbers so that $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 0$. Show that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

p5. Is it possible that a four-digit number $AABB$ is a perfect square?
(Same letters denote the same digits).

p6. A finite number of arcs of a circle are painted black (see figure). The total length of these arcs is less than $\frac{1}{5}$ of the circumference. Show that it is possible to inscribe a square in the circle so that all vertices of the square are in the unpainted portion of the circle.
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png

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2009 **p1.** Arrange the whole numbers 1 through 15 in a row so that the sum of any two adjacent numbers is a perfect square. In how many ways this can be done?

p2. Prove that if p and q are prime numbers which are greater than 3 then $p^2 - q^2$ is divisible by 24.

p3. If a polyleg has even number of legs he always tells truth. If he has an odd number of legs he always lies.

Once a green polyleg told a dark-blue polyleg ”- I have 8 legs. And you have only 6 legs!”
The offended dark-blue polyleg replied ”-It is me who has 8 legs, and you have only 7 legs!”
A violet polyleg added ”-The dark-blue polyleg indeed has 8 legs. But I have 9 legs!”
Then a stripped polyleg started ”None of you has 8 legs. Only I have 8 legs!”
Which polyleg has exactly 8 legs?

[b]p4.[/b] There is a small puncture (a point) in the wall (as shown in the figure below to the right). The housekeeper has a small flag of the following form (see the figure left). Show on the figure all the points of the wall where you can hammer in a nail such that if you hang the flag it will close up the puncture.
`https://cdn.artofproblemsolving.com/attachments/a/f/8bb55a3fdfb0aff8e62bc4cf20a2d3436f5d9`
`png`

p5. Assume a, b, c are odd integers. Show that the quadratic equation $ax^2 + bx + c = 0$ has no rational solutions.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2010 **p1.** Find the smallest whole number $n \geq 2$ such that the product $(2^2 - 1)(3^2 - 1)...(n^2 - 1)$ is the square of a whole number.

p2. The figure below shows a 10×10 square with small 2×2 squares removed from the corners. What is the area of the shaded region?
`https://cdn.artofproblemsolving.com/attachments/7/5/a829487cc5d937060e8965f6da3f4744ba558`
`png`

p3. Three cars are racing: a Ford $[F]$, a Toyota $[T]$, and a Honda $[H]$. They began the race with F first, then T, and H last. During the race, F was passed a total of 3 times, T was passed 5 times, and H was passed 8 times. In what order did the cars finish?

p4. There are 11 big boxes. Each one is either empty or contains 8 medium-sized boxes inside. Each medium box is either empty or contains 8 small boxes inside. All small boxes are empty. Among all the boxes, there are a total of 102 empty boxes. How many boxes are there altogether?

p5. Ann, Mary, Pete, and finally Vlad eat ice cream from a tub, in order, one after another. Each eats at a constant rate, each at his or her own rate. Each eats for exactly the period of time that it would take the three remaining people, eating together, to consume half of the tub. After Vlad eats his portion there is no more ice cream in the tube. How many times faster would it take them to consume the tub if they all ate together?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2012 **p1.** We say that integers a and b are *friends* if their product is a perfect square. Prove that if a is a friend of b, then a is a friend of $gcd(a, b)$.

p2. On the island of knights and liars, a traveler visited his friend, a knight, and saw him sitting at a round table with five guests.
”I wonder how many knights are among you?” he asked.
” Ask everyone a question and find out yourself” advised him one of the guests.
”Okay. Tell me one: Who are your neighbors?” asked the traveler.
This question was answered the same way by all the guests.
”This information is not enough!” said the traveler.
”But today is my birthday, do not forget it!” said one of the guests.
”Yes, today is his birthday!” said his neighbor.
Now the traveler was able to find out how many knights were at the table.
Indeed, how many of them were there if *knights always tell the truth and liars always lie*?

p3. A rope is folded in half, then in half again, then in half yet again. Then all the layers of the rope were cut in the same place. What is the length of the rope if you know that one of the pieces obtained has length of 9 meters and another has length 4 meters?

p4. The floor plan of the palace of the Shah is a square of dimensions 6×6, divided into rooms of dimensions 1×1. In the middle of each wall between rooms is a door. The Shah orders his architect to eliminate some of the walls so that all rooms have dimensions 2×1, no new doors are created, and a path between any two rooms has no more than N doors. What is the smallest value of N such that the order could be executed?

p5. There are 10 consecutive positive integers written on a blackboard. One number is erased. The sum of remaining nine integers is 2011. Which number was erased?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2013 **p1.** A straight line is painted in two colors. Prove that there are three points of the same color such that one of them is located exactly at the midpoint of the interval bounded by the other two.

p2. Find all positive integral solutions x, y of the equation $xy = x + y + 3$.

p3. Can one cut a square into isosceles triangles with angle 80^o between equal sides?

p4. 20 children are grouped into 10 pairs: one boy and one girl in each pair. In each pair the boy is taller than the girl. Later they are divided into pairs in a different way. May it happen now that
(a) in all pairs the girl is taller than the boy;
(b) in 9 pairs out of 10 the girl is taller than the boy?

p5. Mr Mouse got to the cellar where he noticed three heads of cheese weighing 50 grams, 80 grams, and 120 grams. Mr. Mouse is allowed to cut simultaneously 10 grams from any two of the heads and eat them. He can repeat this procedure as many times as he wants. Can he make the weights of all three pieces equal?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2014 **p1.** (a) Put the numbers 1 to 6 on the circle in such way that for any five consecutive numbers the sum of first three (clockwise) is larger than the sum of remaining two.
(b) Can you arrange these numbers so it works both clockwise and counterclockwise.

p2. A girl has a box with 1000 candies. Outside the box there is an infinite number of chocolates and muffins. A girl may replace: • two candies in the box with one chocolate bar, • two muffins in the box with one chocolate bar, • two chocolate bars in the box with one candy and one muffin, • one candy and one chocolate bar in the box with one muffin, • one muffin and one chocolate bar in the box with one candy.
Is it possible that after some time it remains only one object in the box?

p3. Find any integer solution of the puzzle: $WE + ST + RO + NG = 128$ (different letters mean different digits between 1 and 9).

p4. Two consecutive three-digit positive integer numbers are written one after the other one. Show that the six-digit number that is obtained is not divisible by 1001.

p5. There are 9 straight lines drawn in the plane. Some of them are parallel some of them intersect each other. No three lines do intersect at one point. Is it possible to have exactly 17

intersection points?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2015 **p1.** Thirty players participate in a chess tournament. Every player plays one game with every other player. What maximal number of players can get exactly 5 points? (any game adds 1 point to the winner's score, 0 points to a loser's score, in the case of a draw each player obtains $1/2$ point.)

p2. A father and his son returned from a fishing trip. To make their catches equal the father gave to his son some of his fish. If, instead, the son had given his father the same number of fish, then father would have had twice as many fish as his son. What percent more is the father's catch more than his son's?

p3. What is the maximal number of pieces of two shapes, https://cdn.artofproblemsolving.com/attachments/a/5/6c567cf6a04b0aa9e998dbae3803b6eeb24a35.png and https://cdn.artofproble com/attachments/8/a/7a7754d0f2517c93c5bb931fb7b5ae8f5e3217.png, that can be used to tile a 7×7 square?

p4. Six shooters participate in a shooting competition. Every participant has 5 shots. Each shot adds from 1 to 10 points to shooter's score. Every person can score totally for all five shots from 5 to 50 points. Each participant gets 7 points for at least one of his shots. The scores of all participants are different. We enumerate the shooters 1 to 6 according to their scores, the person with maximal score obtains number 1, the next one obtains number 2, the person with minimal score obtains number 6. What score does obtain the participant number 3? The total number of all obtained points is 264.

p5. There are 2014 stones in a pile. Two players play the following game. First, player A takes some number of stones (from 1 to 30) from the pile, then player B takes 1 or 2 stones, then player A takes 2 or 3 stones, then player B takes 3 or 4 stones, then player A takes 4 or 5 stones, etc. The player who gets the last stone is the winner. If no player gets the last stone (there is at least one stone in the pile but the next move is not allowed) then the game results in a draw. Who wins the game using the right strategy?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2017 **p1.** There are 5 weights of masses $1, 2, 3, 5$, and 10 grams. One of the weights is counterfeit (its weight is different from what is written, it is unknown if the weight is heavier or lighter). How to find the counterfeit weight using simple balance scales only twice?

p2. There are 998 candies and chocolate bars and 499 bags. Each bag may contain two items (either two candies, or two chocolate bars, or one candy and one chocolate bar). Ann distributed candies and chocolate bars in such a way that half of the candies share a bag with a chocolate bar. Helen wants to redistribute items in the same bags in such a way that half of the chocolate bars would share a bag with a candy. Is it possible to achieve that?

p3. Insert in sequence 2222222222 arithmetic operations and brackets to get the number 999 (For instance, from the sequence 22222 one can get the number 45: $22 * 2 + 2/2 = 45$).

p4. Put numbers from 15 to 23 in a 3×3 table in such a way to make all sums of numbers in two neighboring cells distinct (neighboring cells share one common side).

p5. All integers from 1 to 200 are colored in white and black colors. Integers 1 and 200 are black, 11 and 20 are white. Prove that there are two black and two white numbers whose sums are equal.

p6. Show that 38 is the sum of few positive integers (not necessarily, distinct), the sum of whose reciprocals is equal to 1. (For instance, $11 = 6 + 3 + 2$, $1/16 + 1/13 + 1/12 = 1$.)

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2018 **p1.** Is it possible to put 9 numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ in a circle in a way such that the sum of any three circularly consecutive numbers is divisible by 3 and is, moreover:
a) greater than 9 ?
b) greater than 15?

p2. You can cut the figure below along the sides of the small squares into several (at least two) identical pieces. What is the minimal number of such equal pieces?
`https://cdn.artofproblemsolving.com/attachments/8/e/9cd09a04209774dab34bc7f989b79573453f3`
`png`

p3. There are 100 colored marbles in a box. It is known that among any set of ten marbles there are at least two marbles of the same color. Show that the box contains 12 marbles of the same color.

p4. Is it possible to color squares of a 8×8 board in white and black color in such a way that every square has exactly one black neighbor square separated by a side?

p5. In a basket, there are more than 80 but no more than 200 white, yellow, black, and red balls. Exactly 12% are yellow, 20% are black. Is it possible that exactly $2/3$ of the balls are white?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2019 **p1.** Prove that the equation $x^6 - 143x^5 - 917x^4 + 51x^3 + 77x^2 + 291x + 1575 = 0$ has no integer solutions.

p2. There are 81 wheels in a storage marked by their two types, say first and second type. Wheels of the same type weigh equally. Any wheel of the second type is much lighter than a wheel of the first type. It is known that exactly one wheel is marked incorrectly. Show that it can be detected with certainty after four measurements on a balance scale.

p3. Rob and Ann multiplied the numbers from 1 to 100 and calculated the sum of digits of this product. For this sum, Rob calculated the sum of its digits as well. Then Ann kept repeating this operation until he got a one-digit number. What was this number?

p4. Rui and Jui take turns placing bishops on the squares of the 8×8 chessboard in such a way that bishops cannot attack one another. (In this game, the color of the rooks is irrelevant.) The player who cannot place a rook loses the game. Rui takes the first turn. Who has a winning strategy, and what is it?

p5. The following figure can be cut along sides of small squares into several (more than one) identical shapes. What is the smallest number of such identical shapes you can get?
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png

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2022 **p1.** Find the unknown angle a of the triangle inscribed in the square.
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png

p2. Draw a polygon in the plane and a point outside of it with the following property: no edge of the polygon is completely visible from that point (in other words, the view is obstructed by some other edge).

p3. This problem has two parts. In each part, 2022 real numbers are given, with some additional property.
(a) Suppose that the sum of any three of the given numbers is an integer. Show that the total sum of the 2022 numbers is also an integer.
(b) Suppose that the sum of any five of the given numbers is an integer. Show that 5 times the total sum of the 2022 numbers is also an integer, but the sum itself is not necessarily an integer.

p4. Replace stars with digits so that the long multiplication in the example below is correct.
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png

p5. Five nodes of a square grid paper are marked (called marked points). Show that there are at least two marked points such that the middle point of the interval connecting them is also a node of the square grid paper

p6. Solve the system

$$\begin{cases} \dfrac{xy}{x+y}=\dfrac{8}{3} \\ \dfrac{yz}{y+z}=\dfrac{12}{5} \\ \dfrac{xz}{x+z}=\dfrac{24}{7} \end{cases}$$

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2023 **p1.** Three camps are located in the vertices of an equilateral triangle. The roads connecting camps are along the sides of the triangle. Captain America is inside the triangle and he needs to know the distances between camps. Being able to see the roads he has found that the sum of the shortest distances from his location to the roads is 50 miles. Can you help Captain America to evaluate the distances between the camps?

p2. N regions are located in the plane, every pair of them have a non-empty overlap. Each region is a connected set, that means every two points inside the region can be connected by a curve all points of which belong to the region. Iron Man has one charge remaining to make a laser shot. Is it possible for him to make the shot that goes through all N regions?

p3. Money in Wonderland comes in $\$5$ and $\$7$ bills.
(a) What is the smallest amount of money you need to buy a slice of pizza that costs $\$1$ and get back your change in full? (The pizza man has plenty of $\$5$ and $\$7$ bills.) For example, having $\$7$ won't do since the pizza man can only give you $\$5$ back.
(b) Vending machines in Wonderland accept only exact payment (do not give back change). List all positive integer numbers which CANNOT be used as prices in such vending machines. (That is, find the sums of money that cannot be paid by exact change.)

p4. (a) Put 5 points on the plane so that each 3 of them are vertices of an isosceles triangle (i.e., a triangle with two equal sides), and no three points lie on the same line.
(b) Do the same with 6 points.

p5. Numbers $1, 2, 3, \dots, 100$ are randomly divided in two groups 50 numbers in each. In the first group the numbers are written in increasing order and denoted $a_1, a_2, ..., a_{50}$. In the second group the numberss are written in decreasing order and denoted $b_1, b_2, ..., b_{50}$. Thus $a_1 < a_2 < ... < a_{50}$ and $b_1 > b_2 > ... > b_{50}$. Evaluate $|a_1 - b_1| + |a_2 - b_2| + ... + |a_{50} - b_{50}|$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).