India National Olympiad 2022

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1 Let D be an interior point on the side BC of an acute-angled triangle ABC. Let the circumcircle of triangle ADB intersect AC again at $E(\neq A)$ and the circumcircle of triangle ADC intersect AB again at $F(\neq A)$. Let AD, BE, and CF intersect the circumcircle of triangle ABC again at $D_1(\neq A)$, $E_1(\neq B)$ and $F_1(\neq C)$, respectively. Let I and I_1 be the incentres of triangles DEF and $D_1E_1F_1$, respectively. Prove that E, F, I, I_1 are concyclic.

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2 Find all natural numbers n for which there is a permutation σ of $\{1, 2, \ldots, n\}$ that satisfies:

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} = 0$$

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3 For a positive integer N, let $T(N)$ denote the number of arrangements of the integers $1, 2, \cdots N$ into a sequence $a_1, a_2, \cdots a_N$ such that $a_i > a_{2i}$ for all i, $1 \leq i < 2i \leq N$ and $a_i > a_{2i+1}$ for all i, $1 \leq i < 2i + 1 \leq N$. For example, $T(3)$ is 2, since the possible arrangements are 321 and 312

(a) Find $T(7)$
(b) If K is the largest non-negative integer so that 2^K divides $T(2^n - 1)$, show that $K = 2^n - n - 1$.
(c) Find the largest non-negative integer K so that 2^K divides $T(2^n + 1)$
