IMC 2014

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–	Day 1
1	Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\mathrm{trace}(M) = a$ and $\det(M) = b$. (Proposed by Stephan Wagner, Stellenbosch University)
2	Consider the following sequence $$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots)$$ Find all pairs (α, β) of positive real numbers such that $\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} a_k}{n^{\alpha}} = \beta$. (Proposed by Tomas Barta, Charles University, Prague)
3	Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \dots, a_n$ such that for each choice of signs the polynomial $$\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \dots \pm a_1 x \pm a_0$$ has n distinct real roots. (Proposed by Stephan Neupert, TUM, Mnchen)
4	Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1 < p_1 < \dots < p_k$. Prove that e_1 is an even number. A number n is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n. (Proposed by Javier Rodrigo, Universidad Pontificia Comillas)
5	Let $A_1A_2 \dots A_{3n}$ be a closed broken line consisting of $3n$ lines segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \dots, 3n$, the triangle $A_iA_{i+1}A_{i+2}$ has counterclockwise orientation and $\angle A_iA_{i+1}A_{i+2} = 60$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$
–	Day 2

1 For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \dots, 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x) 10^{n-1}$. Suppose that for some sequence $(a_n)_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $(d_n(a_n))_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

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2 Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \le i < j \le n} a_{ii} a_{jj} \ge \sum_{1 \le i < j \le n} \lambda_i \lambda_j$$

and determine all matrices for which equality holds.

(Proposed by Matrin Niepel, Comenius University, Bratislava)

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3 Let $f(x) = \frac{\sin x}{x}$, for $x > 0$, and let n be a positive integer. Prove that $|f^{(n)}(x)| < \frac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

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4 We say that a subset of $\mathbb{R}^n$ is k-*almost contained* by a hyperplane if there are less than k points in that set which do not belong to the hyperplane. We call a finite set of points k-*generic* if there is no hyperplane that k-almost contains the set. For each pair of positive integers (k, n), find the minimal number of $d(k, n)$ such that every finite k-generic set in $\mathbb{R}^n$ contains a k-generic subset with at most $d(k, n)$ elements.

(Proposed by Shachar Carmeli, Weizmann Inst. and Lev Radzivilovsky, Tel Aviv Univ.)

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5 For every positive integer n, denote by D_n the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_j \neq j$ for every $1 \le j \le n$. For $1 \le k \le \frac{n}{2}$, denote by $\Delta(n, k)$ the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_i = k + i$ for every $1 \le i \le k$ and $x_j \neq j$ for every $1 \le j \le n$. Prove that

$$\Delta(n, k) = \sum_{i=0}^{k=1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}$$

(Proposed by Combinatorics; Ferdowsi University of Mashhad, Iran; Mirzavaziri)
