Finals 2014

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–	Day 1
1	Let $k, n \geq 1$ be relatively prime integers. All positive integers not greater than $k+n$ are written in some order on the blackboard. We can swap two numbers that differ by k or n as many times as we want. Prove that it is possible to obtain the order $1, 2, \dots, k+n-1, k+n$.
2	Let $k \geq 2$, $n \geq 1$, $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_n$ be integers such that $1 < a_1 < a_2 < \dots < a_k < b_1 < b_2 < \dots < b_n$. Prove that if $a_1+a_2+\dots+a_k > b_1+b_2+\dots+b_n$, then $a_1 \cdot a_2 \cdot \ldots \cdot a_k > b_1 \cdot b_2 \cdot \ldots \cdot b_n$.
3	A tetrahedron $ABCD$ with acute-angled faces is inscribed in a sphere with center O. A line passing through O perpendicular to plane ABC crosses the sphere at point D' that lies on the opposide side of plane ABC than point D. Line DD' crosses plane ABC in point P that lies inside the triangle ABC. Prove, that if $\angle APB = 2\angle ACB$, then $\angle ADD' = \angle BDD'$.
–	Day 2
1	Denote the set of positive rational numbers by $\mathbb{Q}_+$. Find all functions $f: \mathbb{Q}_+ \rightarrow \mathbb{Q}_+$ that satisfy $$\underbrace{f(f(f(\dots f(f}_{n}(q))\dots))) = f(nq)$$ for all integers $n \geq 1$ and rational numbers $q > 0$.
2	Find all pairs (x, y) of positive integers that satisfy $$2^x + 17 = y^4$$.
3	In an acute triangle ABC point D is the point of intersection of altitude h_a and side BC, and points M, N are orthogonal projections of point D on sides AB and AC. Lines MN and AD cross the circumcircle of triangle ABC at points P, Q and A, R. Prove that point D is the center of the incircle of PQR.