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– TST 1 Day 1

1 Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

(Vietnam)

2 Alice has a map of Wonderland, a country consisting of $n \geq 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be ”one way” only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

3 Let a be a positive integer. We say that a positive integer b is [i]a-good[/i] if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a-good, but $b + 2$ is not a-good. Prove that $b + 1$ is prime.

– TST 1 Day 2

4 For any $h = 2^r$ (r is a non-negative integer), find all $k \in \mathbb{N}$ which satisfy the following condition: There exists an odd natural number $m > 1$ and $n \in \mathbb{N}$, such that $k \mid m^h - 1, m \mid n^{\frac{m^h-1}{k}} + 1$.

5 Let x, y, z be nonnegative real numbers such that $x + y + z = 3$. Prove that

$$\frac{x}{4-y} + \frac{y}{4-z} + \frac{z}{4-x} + \frac{1}{16}(1-x)^2(1-y)^2(1-z)^2 \leq 1,$$

and determine all such triples (x, y, z) where the equality holds.

6 Let $n > 1$ be an integer. Suppose we are given $2n$ points in the plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \dots, A_{2n}$ in some order. We then consider the

$2n$ angles $\angle A_1A_2A_3, \angle A_2A_3A_4, \dots, \angle A_{2n-2}A_{2n-1}A_{2n}, \angle A_{2n-1}A_{2n}A_1, \angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

– TST 2 Day 1

1 You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leq r \leq 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

2 Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

(Australia)

3 Let $x_1, x_2, \dots, x_n$ be different real numbers. Prove that

$$\sum_{1\leqslant i\leqslant n}\prod_{j\neq i}\frac{1-x_ix_j}{x_i-x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

– TST 2 Day 2

4 Let n be a positive integer and let P be the set of monic polynomials of degree n with complex coefficients. Find the value of

$$\min_{p\in P}\left\{\max_{|z|=1}|p(z)|\right\}$$

5 Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > 2019$.

6 Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

(Slovakia)

– TST 3 Day 1

1 The infinite sequence $a_0, a_1, a_2, \dots$ of (not necessarily distinct) integers has the following properties: $0 \le a_i \le i$ for all integers $i \ge 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \ge 0$. Prove that all integers $N \ge 0$ occur in the sequence (that is, for all $N \ge 0$, there exists $i \ge 0$ with $a_i = N$).

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2 Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \ne 2 \cdot \angle ADB$.
Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$.
Prove that P lies on the diagonal CE if and only if area (BCD) + area (ADE) = area (ABD) + area (ABP).

(Hungary)

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3 Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)

Russia

– TST 3 Day 2

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4 Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

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5 Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \dots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \{(i, j) \,|\, 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

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6 There are 60 empty boxes $B_1, \dots, B_{60}$ in a row on a table and an unlimited supply of pebbles.
Given a positive integer n, Alice and Bob play the following game.
In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes.
Each subsequent round consists of two steps:

(a) Bob chooses an integer k with $1 \leq k \leq 59$ and splits the boxes into the two groups $B_1, \dots, B_k$ and $B_{k+1}, \dots, B_{60}$.
(b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.
Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

Czech Republic

– TST 4 Day 1

1 Let ABC be a triangle with circumcircle Γ. Let ω_0 be a circle tangent to chord AB and arc ACB. For each $i = 1, 2$, let ω_i be a circle tangent to AB at T_i , to ω_0 at S_i , and to arc ACB. Suppose $\omega_1 \neq \omega_2$. Prove that there is a circle passing through S_1, S_2, T_1, and T_2, and tangent to Γ if and only if $\angle ACB = 90^o$.
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2 On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each $n,$ what are all possible values for $k(\mathfrak{L}),$ where $\mathfrak{L}$ is a labyrinth with n walls?

3 Let $\mathbb{Z}$ be the set of integers. We consider functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\left(f(x+y)+y\right) = f\left(f(x)+y\right)$$

for all integers x and y. For such a function, we say that an integer v is *f-rare* if the set

$$X_v = \{x \in \mathbb{Z} : f(x) = v\}$$

is finite and nonempty.
(a) Prove that there exists such a function f for which there is an f-rare integer.
(b) Prove that no such function f can have more than one f-rare integer.

Netherlands

– TST 4 Day 2

4 Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

5 We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

6 Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1), f(\ell_2), f(\ell_3)$, and X lie on a circle.
Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.

Australia