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– Day 1

1 Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

2 In a classroom of at least four students, when any four of them take seats around a round table, there is always someone who either knows both of his neighbors, or does not know either of his neighbors. Prove that it is possible to divide the students into two groups so that in one of them, all students knows one another, and in the other, none of the students know each other.

[i]Note: If A knows B, then B knows A as well.[/i]

3 Determine all polynomials $P(x,y), Q(x,y)$ and $R(x,y)$ with real coefficients satisfying

$$P(ux+vy, uy+vx) = Q(x,y)R(u,v)$$

for all real numbers u, v, x and y.

– Day 2

1 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

2 Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

3 Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n)|f(m)+f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

– Day 3

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1 Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, ..., n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

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2 Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

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3 Let ABC be any triangle with $\angle BAC \leq \angle ACB \leq \angle CBA$. Let D, E and F be the midpoints of BC, CA and AB, respectively, and let ϵ be a positive real number. Suppose there is an ant (represented by a point T) and two spiders (represented by points P_1 and P_2, respectively) walking on the sides BC, CA, AB, EF, FD and DE. The ant and the spiders may vary their speeds, turn at an intersection point, stand still, or turn back at any point; moreover, they are aware of their and the others' positions at all time.
Assume that the ant's speed does not exceed 1 mm/s, the first spider's speed does not exceed $\frac{\sin A}{2\sin A+\sin B}$ mm/s, and the second spider's speed does not exceed ϵ mm/s. Show that the spiders always have a strategy to catch the ant regardless of the starting points of the ant and the spiders.

Note: the two spiders can discuss a plan before the hunt starts and after seeing all three starting points, but cannot communicate during the hunt.

– Day 4

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1 There are 2^{2018} positions on a circle numbered from 1 to 2^{2018} in a clockwise manner. Initially, two white marbles are placed at positions 2018 and 2019. Before the game starts, Ping chooses to place either a black marble or a white marble at each remaining position. At the start of the game, Ping is given an integer n ($0 \leq n \leq 2018$) and two marbles, one black and one white. He will then move around the circle, starting at position $2n$ and moving clockwise by $2n$ positions at a time. At the starting position and each position he reaches, Ping must switch the marble at that position with a marble of the other color he carries. If he cannot do so at any position, he loses the game. Is there a way to place the $2^{2018} - 2$ remaining marbles so that Ping will never lose the game regardless of the number n and the number of rounds he moves around the circle?

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2 A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Proposed by Mongolia

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3 Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a+b+c+d=100$.

Proposed by Evan Chen, Taiwan

– Day 5

1 Let $n>1$ be a positive integer. Each cell of an $n\times n$ table contains an integer. Suppose that the following conditions are satisfied:

- Each number in the table is congruent to 1 modulo n.
- The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the number in the j^{th} column. Prove that the sums $R_1 + \dots R_n$ and $C_1 + \dots C_n$ are congruent modulo n^4.

2 Determine all functions $f:(0,\infty)\to\mathbb{R}$ satisfying

$$\left(x+\frac{1}{x}\right)f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

3 Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

– Day 6

1 Let n be a positive integer. Let S be a set of n positive integers such that the greatest common divisors of all nonempty sets of S are distinct. Determine the smallest possible number of distinct prime divisors of the product of the elements of S.

2 Let $n\geq 3$ be an integer. Two players play a game on an empty graph with $n+1$ vertices, consisting of the vertices of a regular n-gon and its center. They alternately select a vertex of the n-gon and draw an edge (that has not been drawn) to an adjacent vertex on the n-gon or to the center of the n-gon. The player who first makes the graph connected wins. Between the player who goes first and the player who goes second, who has a winning strategy?

Note: an empty graph is a graph with no edges.

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3 Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

– Day 7

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1 Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

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2 A circle ω with radius 1 is given. A collection T of triangles is called *good*, if the following conditions hold:

- each triangle from T is inscribed in ω;
- no two triangles from T have a common interior point.

Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

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3 Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

- If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.
- If no such pair exists, we write two times the number 0.

Prove that, no matter how we make the choices in (i), operation (ii) will be performed only finitely many times.

Proposed by *Serbia*.

– Day 8

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1 In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Proposed by Iman Maghsoudi

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2 Let $a_0, a_1, a_2, \dots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximum possible value of $a_{2018} - a_{2017}$.

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3 Let $n \geq 2018$ be an integer, and let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

– Day 9

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1 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

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2 Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

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3 Let $m, n \geq 2$ be integers. Let $f(x_1, \dots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \dots, x_n) = \left\lfloor \frac{x_1 + \cdots + x_n}{m} \right\rfloor \text{ for every } x_1, \dots, x_n \in \{0, 1, \dots, m-1\}.$$

Prove that the total degree of f is at least n.

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