VII Caucasus Mathematical Olympiad

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–	Juniors
–	Day 1
1	Positive integers a, b, c are given. It is known that $\frac{c}{b} = \frac{b}{a}$, and the number $b^2 - a - c + 1$ is a prime. Prove that a and c are double of a squares of positive integers.
2	In parallelogram $ABCD$, points E and F on segments AD and CD are such that $\angle BCE = \angle BAF$. Points K and L on segments AD and CD are such that $AK = ED$ and $CL = FD$. Prove that $\angle BKD = \angle BLD$.
3	Pete wrote down 21 pairwise distinct positive integers, each not greater than $1,000,000$. For every pair (a, b) of numbers written down by Pete, Nick wrote the number $$F(a;b) = a + b - \gcd(a;b)$$ on his piece of paper. Prove that one of Nick's numbers differs from all of Pete's numbers.
4	Do there exist 2021 points with integer coordinates on the plane such that the pairwise distances between them are pairwise distinct consecutive integers?
–	Day 2
5	Let S be the set of all 5^6 positive integers, whose decimal representation consists of exactly 6 odd digits. Find the number of solutions (x, y, z) of the equation $x + y = 10z$, where $x \in S$, $y \in S$, $z \in S$.
6	16 NHL teams in the first playoff round divided in pairs and to play series until 4 wins (thus the series could finish with score 4-0, 4-1, 4-2, or 4-3). After that 8 winners of the series play the second playoff round divided into 4 pairs to play series until 4 wins, and so on. After all the final round is over, it happens that k teams have non-negative balance of wins (for example, the team that won in the first round with a score of 4-2 and lost in the second with a score of 4-3 fits the condition: it has $4 + 3 = 7$ wins and $2 + 4 = 6$ losses). Find the least possible k.
7	Point P is chosen on the leg CB of right triangle ABC ($\angle ACB = 90^\circ$). The line AP intersects the circumcircle of ABC at point Q. Let L be the midpoint of PB. Prove that QL is tangent to a fixed circle independent of the choice of point P.

8 Paul can write polynomial $(x+1)^n$, expand and simplify it, and after that change every coefficient by its reciprocal. For example if $n = 3$ Paul gets $(x+1)^3 = x^3 + 3x^2 + 3x + 1$ and then $x^3 + \frac{1}{3}x^2 + \frac{1}{3}x + 1$. Prove that Paul can choose n for which the sum of Paul's polynomial coefficients is less than 2.022.

– Seniors

– Day 1

1 Given a rectangular table with 2 rows and 100 columns. Dima fills the cells of the first row with numbers 1, 2 or 3. Prove that Alex can fill the cells of the second row with numbers 1, 2, 3 in such a way that the numbers in each column are different and the sum of the numbers in the second row equals 200.

2 Prove that infinitely many positive integers can be represented as $(a-1)/b+(b-1)/c+(c-1)/a$, where a, b and c are pairwise distinct positive integers greater than 1.

3 Do there exist 100 points on the plane such that the pairwise distances between them are pairwise distinct consecutive integer numbers larger than 2022?

4 Let ω is tangent to the sides of an acute angle with vertex A at points B and C. Let D be an arbitrary point onn the major arc BC of the circle ω. Points E and F are chosen inside the angle DAC so that quadrilaterals $ABDF$ and $ACED$ are inscribed and the points A, E, F lie on the same straight line. Prove that lines BE and CF intersectat ω.

– Day 2

5 See Juniors 6

6 Let ABC be an acute triangle. Let P be a point on the circle (ABC), and Q be a point on the segment AC such that $AP \perp BC$ and $BQ \perp AC$. Lot O be the circumcenter of triangle APQ. Find the angle OBC.

7 See Juniors 8

8 There are $n > 2022$ cities in the country. Some pairs of cities are connected with straight two-ways airlines. Call the set of the cities *unlucky*, if it is impossible to color the airlines between them in two colors without monochromatic triangle (i.e. three cities A, B, C with the airlines AB, AC and BC of the same color).

The set containing all the cities is unlucky. Is there always an unlucky set containing exactly 2022 cities?