Turkey Team Selection Test 2022

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Day 1 9 March 2022

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1 Find all pairs of prime numbers (p, q) for which

$$2^p = 2^{q-2} + q!.$$

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2 Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfying $f(x) + f(y) = \left(f(x+y) + \frac{1}{x+y}\right)(1 - xy + f(xy))$ for all $x, y \in \mathbb{Q}^+$.

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3 In a triangle ABC, the incircle centered at I is tangent to the sides BC, AC and AB at D, E and F, respectively. Let X, Y and Z be the feet of the perpendiculars drawn from A, B and C to a line ℓ passing through I. Prove that DX, EY and FZ are concurrent.

Day 2 10 March 2022

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4 We have three circles w_1, w_2 and Γ at the same side of line l such that w_1 and w_2 are tangent to l at K and L and to Γ at M and N, respectively. We know that w_1 and w_2 do not intersect and they are not in the same size. A circle passing through K and L intersect Γ at A and B. Let R and S be the reflections of M and N with respect to l. Prove that A, B, R, S are concyclic.

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5 On a circle, 2022 points are chosen such that distance between two adjacent points is always the same. There are k arcs, each having endpoints on chosen points, with different lengths. Arcs do not contain each other. What is the maximum possible number of k?

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6 For a polynomial $P(x)$ with integer coefficients and a prime p, if there is no $n \in \mathbb{Z}$ such that $p|P(n)$, we say that polynomial P *excludes* p. Is there a polynomial with integer coefficients such that having degree of 5, excluding exactly one prime and not having a rational root?

Day 3 11 March 2022

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7 What is the minimum value of the expression

$$xy + yz + zx + \frac{1}{x} + \frac{2}{y} + \frac{5}{z}$$

where x, y, z are positive real numbers?

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8 ABC triangle with $|AB| < |BC| < |CA|$ has the incenter I. The orthocenters of triangles IBC, IAC and IAB are H_A, H_A and H_A. H_BH_C intersect BC at K_A and perpendicular line from I to H_BH_B intersect BC at L_A. K_B, L_B, K_C, L_C are defined similarly. Prove that

$$|K_AL_A| = |K_BL_B| + |K_CL_C|$$

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9 In every acyclic graph with 2022 vertices we can choose k of the vertices such that every chosen vertex has at most 2 edges to chosen vertices. Find the maximum possible value of k.
