

JBMO TST - Turkey 2022

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Day 1 10 March 2022

1 For positive integers a and b , if the expression $\frac{a^2+b^2}{(a-b)^2}$ is an integer, prove that the expression $\frac{a^3+b^3}{(a-b)^3}$ is an integer as well.

2 For a real number a , $[a]$ denotes the largest integer not exceeding a . Find all positive real numbers x satisfying the equation

$$x \cdot [x] + 2022 = [x^2]$$

3 Each of the 29 people attending a party wears one of three different types of hats. Call a person *lucky* if at least two of his friends wear different types of hats. Show that it is always possible to replace the hat of a person at this party with a hat of one of the other two types, in a way that the total number of lucky people is not reduced.

4 Given a convex quadrilateral $ABCD$ such that $m(\widehat{ABC}) = m(\widehat{BCD})$. The lines AD and BC intersect at a point P and the line passing through P which is parallel to AB , intersects BD at T . Prove that

$$m(\widehat{ACB}) = m(\widehat{PCT})$$

Day 2 11 March 2022

5 Each of the n students writes one of the numbers 1, 2 or 3 on each of the 29 boards. If any two students wrote different numbers on at least one of the boards and any three students wrote the same number on at least one of the boards, what is the maximum possible value of n ?

6 Let c be a real number. If the inequality

$$f(c) \cdot f(-c) \geq f(a)$$

holds for all $f(x) = x^2 - 2ax + b$ where a and b are arbitrary real numbers, find all possible values of c .

- 7 In a triangle $\triangle ABC$ with $\angle ABC < \angle BCA$, we define K as the excenter with respect to A . The lines AK and BC intersect in a point D . Let E be the circumcenter of $\triangle BKC$. Prove that

$$\frac{1}{|KA|} = \frac{1}{|KD|} + \frac{1}{|KE|}.$$

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- 8 Find all prime numbers p such that the number

$$3^p + 4^p + 5^p + 9^p - 98$$

has at most 6 positive divisors.
