Mathematical Olympiad 2022

www.artofproblemsolving.com/community/c3012070

by parmenides51, InternetPerson10

– Day 1

1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d)$$

for all real numbers a, b, c, and d.

2 The PMO Magician has a special party game. There are n chairs, labelled 1 to n. There are n sheets of paper, labelled 1 to n.

- On each chair, she attaches exactly one sheet whose number does not match the number on the chair.
- She then asks n party guests to sit on the chairs so that each chair has exactly one occupant.
- Whenever she claps her hands, each guest looks at the number on the sheet attached to their current chair, and moves to the chair labelled with that number.

Show that if $1 < m \leq n$, where m is not a prime power, it is always possible for the PMO Magician to choose which sheet to attach to each chair so that everyone returns to their original seats after exactly m claps.

3 Call a lattice point *visible* if the line segment connecting the point and the origin does not pass through another lattice point. Given a positive integer k, denote by S_k the set of all visible lattice points (x, y) such that $x^2 + y^2 = k^2$. Let D denote the set of all positive divisors of $2021 \cdot 2025$. Compute the sum

$$\sum_{d \in D} |S_d|$$

Here, a lattice point is a point (x, y) on the plane where both x and y are integers, and $|A|$ denotes the number of elements of the set A.

4 Let $\triangle ABC$ have incenter I and centroid G. Suppose that P_A is the foot of the perpendicular from C to the exterior angle bisector of B, and Q_A is the foot of the perpendicular from B to the exterior angle bisector of C. Define P_B, P_C, Q_B, and Q_C similarly. Show that $P_A, P_B, P_C, Q_A, Q_B,$ and Q_C lie on a circle whose center is on line IG.

– Day 2

5 Find all positive integers n for which there exists a set of exactly n distinct positive integers, none of which exceed n^2, whose reciprocals add up to 1.

6 In $\triangle ABC$, let D be the point on side BC such that $AB+BD = DC+CA$. The line AD intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. Prove that one of the common tangents of the circumcircles of $\triangle BDX$ and $\triangle CDX$ is parallel to BC.

7 Let $a, b,$ and c be positive real numbers such that $ab + bc + ca = 3$. Show that

$$\frac{bc}{1+a^4} + \frac{ca}{1+b^4} + \frac{ab}{1+c^4} \geq \frac{3}{2}.$$

8 The set $S = \{1, 2, \ldots, 2022\}$ is to be partitioned into n disjoint subsets $S_1, S_2, \ldots, S_n$ such that for each $i \in \{1, 2, \ldots, n\}$, exactly one of the following statements is true:

(a) For all $x, y \in S_i$, with $x \neq y, \gcd(x, y) > 1.$
(b) For all $x, y \in S_i$, with $x \neq y, \gcd(x, y) = 1.$

Find the smallest value of n for which this is possible.