Canada National Olympiad 2022

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1 Assume that real numbers a and b satisfy

$$ab + \sqrt{ab+1} + \sqrt{a^2+b}\sqrt{a+b^2} = 0.$$

Find, with proof, the value of

$$b\sqrt{a^2+b} + a\sqrt{b^2+a}.$$

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2 Let $d(k)$ denote the number of positive integer divisors of k. For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, $1, 2, 3$, and 6. Prove that for all positive integers n,

$$d(1) + d(3) + d(5) + ... + d(2n-1) \leq d(2) + d(4) + d(6) + ... + d(2n).$$

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3 Vishal starts with n copies of the number 1 written on the board. Every minute, he takes two numbers a, b and replaces them with either $a+b$ or $\min(a^2, b^2)$. After $n-1$ there is 1 number on the board. Let the maximal possible value of this number be $f(n)$. Prove $2^{n/3} < f(n) \leq 3^{n/3}$.

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4 Let n be a positive integer. A set of n distinct lines divides the plane into various (possibly unbounded) regions. The set of lines is called ”nice” if no three lines intersect at a single point. A ”colouring” is an assignment of two colours to each region such that the first colour is from the set $\{A_1, A_2\}$, and the second colour is from the set $\{B_1, B_2, B_3\}$. Given a nice set of lines, we call it ”colourable” if there exists a colouring such that
(a) no colour is assigned to two regions that share an edge;
(b) for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ there is at least one region that is assigned with both A_i and B_j .
Determine all n such that every nice configuration of n lines is colourable.

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5 Let $ABCDE$ be a convex pentagon such that the five vertices lie on a circle and the five sides are tangent to another circle inside the pentagon. There are $\binom{5}{3} = 10$ triangles which can be formed by choosing 3 of the 5 vertices. For each of these 10 triangles, mark its incenter. Prove that these 10 incenters lie on two concentric circles.
