Problems from the CMIMC 2022

www.artofproblemsolving.com/community/c3012505

by parmenides51, AOPS12142015

– Algebra & Number Theory

2.1 Alice and Bob live on the same road. At time t, they both decide to walk to each other's houses at constant speed. However, they were busy thinking about math so that they didn't realize passing each other. Alice arrived at Bob's house at $3:19$pm, and Bob arrived at Alice's house at $3:29$pm. Charlie, who was driving by, noted that Alice and Bob passed each other at $3:11$pm. Find the difference in minutes between the time Alice and Bob left their own houses and noon on that day.

Proposed by Kevin You

2.2 Arthur, Bob, and Carla each choose a three-digit number. They each multiply the digits of their own numbers. Arthur gets 64, Bob gets 35, and Carla gets 81. Then, they add corresponding digits of their numbers together. The total of the hundreds place is 24, that of the tens place is 12, and that of the ones place is 6. What is the difference between the largest and smallest of the three original numbers?

Proposed by Jacob Weiner

2.3 1.1 How many 4-digit numbers have exactly 9 divisors from the set $\{1,2,3,4,5,6,7,8,9,10\}$?

Proposed by Ethan Gu

2.4 1.2 A shipping company charges $.30l+.40w+.50h$ dollars to process a right rectangular prism-shaped box with dimensions l,w,h in inches. The customers themselves are allowed to label the three dimensions of their box with l,w,h for the purpose of calculating the processing fee. A customer finds that there are two different ways to label the dimensions of their box B to get a fee of $\$8.10$, and two different ways to label B to get a fee of $\$8.70$. None of the faces of B are squares. Find the surface area of B, in square inches.

Proposed by Justin Hsieh

2.5 Alan is assigning values to lattice points on the 3d coordinate plane. First, Alan computes the roots of the cubic $20x^3-22x^2+2x+1$ and finds that they are α, β, and γ. He finds out that each of these roots satisfy $|\alpha|,|\beta|,|\gamma|\leq 1$ On each point (x,y,z) where $x,y,$ and z are all nonnegative integers, Alan writes down $\alpha^x\beta^y\gamma^z$. What is the value of the sum of all numbers he writes down?

Proposed by Alan Abraham

2.6 1.3 Find the smallest positive integer N such that each of the 101 intervals

$$[N^2,(N+1)^2),[(N+1)^2,(N+2)^2),\cdots,[(N+100)^2,(N+101)^2)$$

contains at least one multiple of 1001.

Proposed by Kyle Lee

2.7 For polynomials $P(x)=a_nx^n+\cdots+a_0$, let $f(P)=a_n\cdots a_0$ be the product of the coefficients of P. The polynomials P_1,P_2,P_3,Q satisfy $P_1(x)=(x-a)(x-b)$, $P_2(x)=(x-a)(x-c)$, $P_3(x)=(x-b)(x-c)$, $Q(x)=(x-a)(x-b)(x-c)$ for some complex numbers a,b,c. Given $f(Q)=8$, $f(P_1)+f(P_2)+f(P_3)=10$, and $abc>0$, find the value of $f(P_1)f(P_2)f(P_3)$.

Proposed by Justin Hsieh

2.8 1.4 Let z be a complex number that satisfies the equation

$$\frac{z-4}{z^2-5z+1}+\frac{2z-4}{2z^2-5z+1}+\frac{z-2}{z^2-3z+1}=\frac{3}{z}.$$

Over all possible values of z, find the sum of the values of

$$\left|\frac{1}{z^2-5z+1}+\frac{1}{2z^2-5z+1}+\frac{1}{z^2-3z+1}\right|.$$

Proposed by Justin Hsieh

1.5 Grant is standing at the beginning of a hallway with infinitely many lockers, numbered, $1,2,3,\ldots$ All of the lockers are initially closed. Initially, he has some set $S=\{1,2,3,\ldots\}$.

Every step, for each element s of S, Grant goes through the hallway and opens each locker divisible by s that is closed, and closes each locker divisible by s that is open. Once he does this for all s, he then replaces S with the set of labels of the currently open lockers, and then closes every door again.

After 2022 steps, S has n integers that divide 10^{2022}. Find n.

Proposed by Oliver Hayman

1.6 Find the probability such that when a polynomial in $\mathbb{Z}_{2027}[x]$ having degree at most 2026 is chosen uniformly at random,

$$x^{2027}-x|P^k(x)-x \iff 2021|k$$

(note that 2027 is prime).

Here $P^k(x)$ denotes P composed with itself k times.

Proposed by Grant Yu

1.7 Let $f(n)$ count the number of values $0 \le k \le n^2$ such that $43 \nmid \binom{n^2}{k}$. Find the least positive value of n such that

$$43^{43} \mid f\left(\frac{43^n - 1}{42}\right)$$

Proposed by Adam Bertelli

1.8 Find the largest $c > 0$ such that for all $n \ge 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$ we have

$$\sum_{j=1}^{n} a_j^4 \ge c \sum_{k=1}^{n} \frac{\left(\sum_{j=1}^{k} a_j b_{k+1-j}\right)^4}{\left(\sum_{j=1}^{k} b_j^2 j!\right)^2}$$

Proposed by Grant Yu

– Geometry

2.1 An equilateral 12-gon has side length 10 and interior angle measures that alternate between 90°, 90°, and 270°. Compute the area of this 12-gon.

Proposed by Connor Gordon

2.2 1.1 A circle has radius 52 and center O. Points A is on the circle, and point P on $\overline{OA}$ satisfies $OP = 28$. Point Q is constructed such that $QA = QP = 15$, and point B is constructed on the circle so that Q is on $\overline{OB}$. Find QB.

Proposed by Justin Hsieh

2.3 1.2 Let ABC be an acute triangle with $\angle ABC = 60^\circ$. Suppose points D and E are on lines AB and CB, respectively, such that CDB and AEB are equilateral triangles. Given that the positive difference between the perimeters of CDB and AEB is 60 and $DE = 45$, what is the value of $AB \cdot BC$?

Proposed by Kyle Lee

2.4 Circle Γ has diameter $\overline{AB}$ with $AB = 6$. Point C is constructed on line AB so that $AB = BC$ and $A \neq C$. Let D be on Γ so that $\overleftrightarrow{CD}$ is tangent to Γ. Compute the distance from line $\overleftrightarrow{AD}$ to the circumcenter of $\triangle ADC$.

Proposed by Justin Hsieh

2.5 Let ABC be an equilateral triangle of unit side length and suppose D is a point on segment $\overline{BC}$ such that $DB < DC$. Let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Suppose X and Y are the intersections of lines AB and ND, and lines AC and MD, respectively. Given that $XY = 4$, what is the value of $\frac{DB}{DC}$?

Proposed by Kyle Lee

2.6 A triangle $\triangle ABC$ satisfies $AB = 13$, $BC = 14$, and $AC = 15$. Inside $\triangle ABC$ are three points X, Y, and Z such that:

- Y is the centroid of $\triangle ABX$
- Z is the centroid of $\triangle BCY$
- X is the centroid of $\triangle CAZ$

What is the area of $\triangle XYZ$?

Proposed by Adam Bertelli

2.7 1.3 Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three pairwise externally tangent circles with radii $1, 2, 3,$ respectively. A circle passes through the centers of Γ_2 and Γ_3 and is externally tangent to Γ_1 at a point P. Suppose A and B are the centers of Γ_2 and $\Gamma_3,$ respectively. What is the value of $\frac{PA^2}{PB^2}$?

Proposed by Kyle Lee

2.8 1.4 Let A and B be points on circle Γ such that $AB = \sqrt{10}$. Point C is outside Γ such that $\triangle ABC$ is equilateral. Let D be a point on Γ and suppose the line through C and D intersects AB and Γ again at points E and $F \neq D$. It is given that points C, D, E, F are collinear in that order and that $CD = DE = EF$. What is the area of Γ?

Proposed by Kyle Lee

1.5 In triangle ABC, let I, O, H be the incenter, circumcenter and orthocenter, respectively. Suppose that $AI = 11$ and $AO = AH = 13$. Find OH.

Proposed by Kevin You

1.6 Let Γ_1 and Γ_2 be two circles with radii r_1 and $r_2,$ respectively, where $r_1 > r_2$. Suppose Γ_1 and Γ_2 intersect at two distinct points A and B. A point C is selected on ray $\overrightarrow{AB}$, past B, and the tangents to Γ_1 and Γ_2 from C are marked as points P and $Q,$ respectively. Suppose that Γ_2 passes through the center of Γ_1 and that points P, B, Q are collinear in that order, with $PB = 3$ and $QB = 2$. What is the length of AB?

Proposed by Kyle Lee

1.7 In acute $\triangle ABC,$ let I denote the incenter and suppose that line AI intersects segment BC at a point D. Given that $AI = 3, ID = 2,$ and $BI^2 + CI^2 = 64,$ compute BC^2.

Proposed by Kyle Lee

1.8 Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Rays $\overrightarrow{OB}$ and $\overrightarrow{DC}$ intersect at E, and rays $\overrightarrow{OC}$ and $\overrightarrow{AB}$ intersect at F. Suppose that $AE = EC = CF = 4$, and the circumcircle of ODE bisects $\overline{BF}$. Find the area of triangle ADF.

Proposed by Howard Halim

– Combinatorics

2.1 A particle starts at $(0, 0, 0)$ in three-dimensional space. Each second, it randomly selects one of the eight lattice points a distance of $\sqrt{3}$ from its current location and moves to that point. What is the probability that, after two seconds, the particle is a distance of $2\sqrt{2}$ from its original location?

Proposed by Connor Gordon

2.2 1.1 Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

Proposed by Nancy Kuang

2.3 We say that a set S of 3 unit squares is *commutable* if $S = \{s_1, s_2, s_3\}$ for some s_1, s_2, s_3 where s_2 shares a side with each of s_1, s_3. How many ways are there to partition a 3×3 grid of unit squares into 3 pairwise disjoint commutable sets?

Proposed by Srinivasan Sathiamurthy

2.4 Dilhan is running around a track for 12 laps. If halfway through a lap, Dilhan has his phone on him, he has a $\frac{1}{3}$ chance to drop it there. If Dilhan runs past his phone on the ground, he will attempt to pick it up with a $\frac{2}{3}$ chance of success, and won't drop it for the rest of the lap. He starts with his phone at the start of the 5K, what is the chance he still has it when he finished the 5K?

Proposed by Zack Lee, Daniel Li, Dilhan Salgado

2.5 Daniel, Ethan, and Zack are playing a multi-round game of Tetris. Whoever wins 11 rounds first is crowned the champion. However Zack is trying to pull off a "reverse-sweep", where (at-least) one of the other two players first hits 10 wins while Zack is still at 0, but Zack still ends up being the first to reach 11. How many possible sequences of round wins can lead to Zack pulling off a reverse sweep?

Proposed by Dilhan Salgado

2.6 1.2 A sequence of pairwise distinct positive integers is called averaging if each term after the first two is the average of the previous two terms. Let M be the maximum possible number of terms in an averaging sequence in which every term is less than or equal to 2022 and let N be the number of such distinct sequences (every term less than or equal to 2022) with exactly M terms. What is $M+N$? (Two sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are said to be distinct if $a_i \neq b_i$ for some integer $1 \leq i \leq n$).

Proposed by Kyle Lee

2.7 1.3 For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them.

Proposed by Nicole Sim

2.8 1.4 The CMU Kiltie Band is attempting to crash a helicopter via grappling hook. The helicopter starts parallel (angle 0 degrees) to the ground. Each time the band members pull the hook, they tilt the helicopter forward by either x or $x+1$ degrees, with equal probability, if the helicopter is currently at an angle x degrees with the ground. Causing the helicopter to tilt to 90 degrees or beyond will crash the helicopter. Find the expected number of times the band must pull the hook in order to crash the helicopter.

Proposed by Justin Hsieh

1.5 At CMIMC headquarters, there is a row of n lightbulbs, each of which is connected to a light switch. Daniel the electrician knows that exactly one of the switches doesn't work, and needs to find out which one. Every second, he can do exactly one of 3 things:

- Flip a switch, changing the lightbulb from off/on or on/off (unless the switch is broken).
- Check if a given lightbulb is on or off.
- Measure the total electricity usage of all the lightbulbs, which tells him exactly how many are currently on.

Initially, all the lightbulbs are off. Daniel was given the very difficult task of finding the broken switch in at most n seconds, but fortunately he showed up to work 10 seconds early today. What is the largest possible value n such that he can complete his task on time?

Proposed by Adam Bertelli

1.6 Barry has a standard die containing the numbers 1-6 on its faces.

He rolls the die continuously, keeping track of the sum of the numbers he has rolled so far, starting from 0. Let E_n be the expected number of time he needs to until his recorded sum is at least n.

It turns out that there exist positive reals a, b such that

$$\lim_{n\to\infty} E_n - (an+b) = 0$$

Find (a, b).

Proposed by Dilhan Salgado

1.7 In a class of 12 students, no two people are the same height. Compute the total number of ways for the students to arrange themselves in a line such that:

- for all $1 < i < 12$, the person in the i-th position (with the leftmost position being 1) is taller than exactly $i \pmod 3$ of their adjacent neighbors, and
- the students standing at positions which are multiples of 3 are strictly increasing in height from left to right.

Proposed by Nancy Kuang

1.8 Daniel has a (mostly) standard deck of 54 cards, consisting of 4 suits each containing the ranks 1 to 13 as well as 2 jokers. Daniel plays the following game: He shuffles the deck uniformly randomly and then takes all of the cards that end up strictly between the two jokers. He then sums up the ranks of all the cards he has taken and calls that his score.

Let p be the probability that his score is a multiple of 13. There exists relatively prime positive integers a and b, with b as small as possible, such that $|p - a/b| < 10^{-10}$. What is a/b?

Proposed by Dilhan Salgado, Daniel Li

– Team

---

1 Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two squares such that the boundaries of $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ does not contain any line segment. Construct 16 line segments A_iB_j for each possible $i, j \in \{1, 2, 3, 4\}$. What is the maximum number of line segments that don't intersect the edges of $A_1A_2A_3A_4$ or $B_1B_2B_3B_4$? (intersection with a vertex is not counted).

Proposed by Allen Zheng

---

2 Find the smallest positive integer n for which $315^2 - n^2$ evenly divides $315^3 - n^3$.

Proposed by Kyle Lee

---

3 Let $ABCD$ be a rectangle with $AB = 10$ and $AD = 5$. Suppose points P and Q are on segments CD and BC, respectively, such that the following conditions hold:

- $BD \parallel PQ$
- $\angle APQ = 90^\circ$.

What is the area of $\triangle CPQ$?

Proposed by Kyle Lee

---

4 Let $\triangle ABC$ be equilateral with integer side length. Point X lies on $\overline{BC}$ strictly between B and C such that $BX < CX$. Let C' denote the reflection of C over the midpoint of $\overline{AX}$. If $BC' = 30$, find the sum of all possible side lengths of $\triangle ABC$.

Proposed by Connor Gordon

---

5 For any integer a, let $f(a) = |a^4 - 36a^2 + 96a - 64|$. What is the sum of all values of $f(a)$ that are prime?

Proposed by Alexander Wang

---

6 There are 9 points arranged in a 3×3 square grid. Let two points be adjacent if the distance between them is half the side length of the grid. (There should be 12 pairs of adjacent points). Suppose that we wanted to connect 8 pairs of adjacent points, such that all points are connected to each other. In how many ways is this possible?

Proposed by Kevin You

---

7 A $3 \times 2 \times 2$ right rectangular prism has one of its edges with length 3 replaced with an edge of length 5 parallel to the original edge. The other 11 edges remain the same length, and the 6 vertices that are not endpoints of the replaced edge remain in place. The resulting convex solid has 8 faces, as shown below.

Find the volume of the solid.

Proposed by Justin Hsieh

---

8 There are 36 contestants in the CMU Puyo-Puyo Tournament, each with distinct skill levels.

The tournament works as follows:
First, all $\binom{36}{2}$ pairings of players are written down on slips of paper and are placed in a hat.
Next, a slip of paper is drawn from the hat, and those two players play a match. It is guaranteed that the player with a higher skill level will always win the match.
We continue drawing slips (without replacement) and playing matches until the results of the match completely determine the order of skill levels of all 36 contestants (i.e. there is only one possible ordering of skill levels consistent with the match results), at which point the tournament immediately finishes.
What is the expected value of the number of matches played before the stopping point is reached?

Proposed by Dilhan Salgado

---

9 For natural numbers n, let $r(n)$ be the number formed by reversing the digits of n, and take $f(n)$ to be the maximum value of $\frac{r(k)}{k}$ across all n-digit positive integers k.

If we define $g(n) = \left\lfloor \frac{1}{10-f(n)} \right\rfloor$, what is the value of $g(20)$?

Proposed by Adam Bertelli

---

10 Adam places down cards one at a time from a standard 52 card deck (without replacement) in a pile. Each time he places a card, he gets points equal to the number of cards in a row immediately before his current card that are all the same suit as the current card. For instance, if there are currently two hearts on the top of the pile (and the third card in the pile is not hearts), then placing a heart would be worth 2 points, and placing a card of any other suit would be worth 0 points. What is the expected number of points Adam will have after placing all 52 cards?

Proposed by Adam Bertelli

---

11 Let $\{\varepsilon_i\}_{i\geq 1}, \{\theta_i\}_{i\geq 0}$ be two infinite sequences of real numbers, such that $\varepsilon_i \in \{-1, 1\}$ for all i, and the numbers θ_i obey

$$\tan\theta_{n+1} = \tan\theta_n + \varepsilon_n \sec(\theta_n) - \tan\theta_{n-1}, \qquad n \geq 1$$

and $\theta_0 = \frac{\pi}{4}, \theta_1 = \frac{2\pi}{3}$. Compute the sum of all possible values of

$$\lim_{m\to\infty}\left(\sum_{n=1}^{m} \frac{1}{\tan\theta_{n+1} + \tan\theta_{n-1}} + \tan\theta_m - \tan\theta_{m+1}\right)$$

Proposed by Grant Yu

---

12 Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 6, DA = 8,$ and circumcircle Γ. The tangents to Γ at A and C intersect at P and the tangents to Γ at B and D intersect at Q. Suppose lines PB and PD intersect Γ at points $W \neq B$ and $X \neq D$, respectively. Similarly, suppose lines QA and QC intersect Γ at points $Y \neq A$ and $Z \neq C$, respectively. What is the value of $\frac{WX^2}{YZ^2}$?

Proposed by Kyle Lee

---

13 Let F_n denote the nth Fibonacci number, with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exists a unique two digit prime p such that for all n, $p | F_{n+100} + F_n$. Find p.

Proposed by Sam Rosenstrauch

---

14 Let a tree on $mn+1$ vertices be (m,n)-nice if the following conditions hold:

- $m+1$ colors are assigned to the nodes of the tree
- for the first m colors, there will be exactly n nodes of color i $(1 \le i \le m)$
- the root node of the tree will be the unique node of color $m+1$.

the (m,n)-nice trees must also satisfy the condition that for any two non-root nodes i,j, if the color of i equals the color of j, then the color of the parent of i equals the color of the parent of j.
- Nodes of the same color are considered indistinguishable (swapping any two of them results in the same tree).

Let $N(u,v,l)$ denote the number of (u,v)-nice trees with l leaves. Note that $N(2,2,2)=2, N(2,2,3)=4, N(2,2,4)=6$. Compute the remainder when $\sum_{l=123}^{789} N(8,101,l)$ is divided by 101.

Definition: Any rooted, ordered tree consists of some set of nodes, each of which has a (possibly empty) ordered list of children. Each node is the child of exactly one other node, with the exception of the root, which has not parent. There also cannot be any cycles of nodes which are all linearly children of each other.

Proposed by Advait Nene

---

15 Let ABC be a triangle with $AB=5, BC=13$, and $AC=12$. Let D be a point on minor arc AC of the circumcircle of ABC (endpoints excluded) and P on $\overline{BC}$. Let B_1, C_1 be the feet of perpendiculars from P onto CD, AB respectively and let BB_1, CC_1 hit (ABC) again at B_2, C_2 respectively. Suppose that D is chosen uniformly at random and AD, BC, B_2C_2 concur at a single point. Compute the expected value of BP/PC.

Proposed by Grant Yu
